

# Various Crosscap Numbers of Knots and Links

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## Abstract

We define the crosscap number of a 2-component link as the minimum of the first Betti numbers of connected, non-orientable surfaces bounding the link. We discuss some basic but important properties of the crosscap numbers of 2-component links.

Then we define the concordance crosscap number  $\gamma_c(K)$  of a knot  $K$  as the minimum crosscap number among all the knots concordant to  $K$ . The four-dimensional crosscap number  $\gamma^*(K)$  is the minimum first Betti number of non-orientable surfaces smoothly embedded in the 4-dimensional ball, bounding the knot  $K$ . Clearly  $\gamma^*(K) \leq \gamma_c(K)$ . We construct two infinite sequences of knots for which  $\gamma^*(K) < \gamma_c(K)$ . In particular, the knot  $7_4$  is one of the examples.

# Chapter 1

## Introduction

We work in the smooth category. A knot in the language of mathematics is an embedding of a circle  $S^1$  into the Euclidean 3-space,  $\mathbf{R}^3$ , or the 3-sphere,  $\mathbf{S}^3$ . If we have an embedding of several disjoint circles, we obtain a link.

Knots can be studied from many points of view. In mathematics they are considered as a part of the geometric topology. Motivation for such a topological study of knots is meant to come from the curiosity to know how the geometry of the three-dimensional space can be explored by knotting phenomena using precise mathematics. The aim is to find invariants that can distinguish knots, to investigate geometric properties of knots and to see something of the way they interact with the more adventurous three-dimensional topology. Knot theory has been proved to be necessary for the study of 3-dimensional manifold. For example, link exteriors give interesting concrete examples of 3-dimensional manifolds, and every closed connected orientable 3-dimensional manifold is obtained by a Dehn surgery manifold along a link and as a branched covering manifold over  $S^3$  with branch set a

knot.

In order to distinguish different knots we define and compare a number (or more generally an algebraic system) which is invariant under the auto-homeomorphisms of  $S^3$ . Such a number (or algebraic system) is called a *knot invariant*. For example, the minimal crossing number of all regular presentations of a knot, the *crossing number*, is probably the easiest intuitive knot invariant. Other famous polynomial knot invariants are *Alexander polynomial*, *Jones polynomial*, the recently popular *Khovanov polynomial* and etc.

It is known that every knot can be considered as the boundary of some surface. As a result, the study of the surfaces becomes an indirect tool to study knots. A compact orientable surface bounded by a knot is called a *Seifert surface* for the knot. Seifert surfaces for a given knot are not uniquely determined, but a knot invariant is obtained from Seifert surfaces, *knot genus* or *orientable genus* due to the orientability of the surfaces, by taking the minimal genus of such Seifert surfaces. The problem to decide the knot genus for a given knot has been completely settled by Ozsvath and Szabo using the knot Heegaard Floer homology. For the compact orientable surfaces embedded in the 4-ball  $B^4$  bounding the given knot, one can also define the *4-dimensional genus* by taking the minimal genus among all such surfaces. One can also define the *concordance genus* to be the minimal genus in its concordance class, namely the knots which can be connected to it by an annulus smoothly embedded in  $S^3 \times I$ . The relations among these different genera have been well studied. We will give a survey on the research results of various knot genera on Chapter 3.

It is not hard to see that the surfaces bounding the knot are not necessarily orientable. In the case that the surfaces are non-orientable, one can define the *crosscap number* of a knot to be the minimal first Betti number among them. This knot invariant was first introduced by Clark in 1978, and his research on this topic was followed by Murakami and Yasuhara in 1995. Recently, the crosscap numbers of several families of knots, like torus knots, two-bridge knots and pretzel knots have been determined by Teragaito, Hirasawa-Teragaito and Ichihara-Mizushima respectively. We will give a survey of all these results in Chapter 4.

The research on the crosscap numbers has been only restricted to knots, links of one component. It is natural to generalize it for links of more than one component. In Chapter 5, we define the crosscap numbers for two-component links and discuss some basic properties of the concept, then also, calculate the crosscap number for the two-component link  $6_3^2$  by using the signature and the linking form.

If the non-orientable surfaces bounding the knot are embedded in the 4-ball instead of the 3-sphere, we can define the *4-dimensional crosscap number* to be the minimal first Betti number of such surfaces. This was originally defined by Murakami and Yasuhara [25]. In Chapter 6, we define the *concordance crosscap number* to be the minimal one in its concordance class, and prove the existence of infinitely many knots to show the difference between the 4-dimensional crosscap number and the concordance one.

The basics of knot theory are introduced in Chapter 2 Preliminaries , where the necessary knowledge of number theory applied in some proof of

later chapters is also listed.

# Chapter 2

## Preliminaries

An (unoriented) link  $L$  of  $m$  components ( $m \geq 1$ ) is a union of  $m$  pairwise disjoint, piecewise linear, simple closed curves in the oriented 3-sphere  $S^3$ . If each component  $L_i$  of a link  $L$  can be assigned a direction going along the circle, the link  $L$  is oriented and called an oriented link. A choice of orientation indicated by arrows on a diagram is extra information given to the link. For each component we have two different ways to orient it, and so in total there are  $2^{m-1}$  different relative orientations for a link of  $m$  components.

A link of one component is usually called *knots*. A knot is said to be the *unknot* if it bounds an embedded piecewise linear disk in  $S^3$ . Two oriented knots  $K_1$  and  $K_2$  can be added together to form the sum, called *connected sum*. Regarding  $K_1$  and  $K_2$  embedded in distinct copies of  $S^3$ , remove from each  $S^3$  a small ball that meets the given knot in an unknotted arc, and then identify together the boundary spheres and their intersection with knots, in a way to match up the orientations. Denote the connected sum of the knots  $K_1$  and  $K_2$  by  $K_1 \sharp K_2$  or  $K_1 + K_2$ .

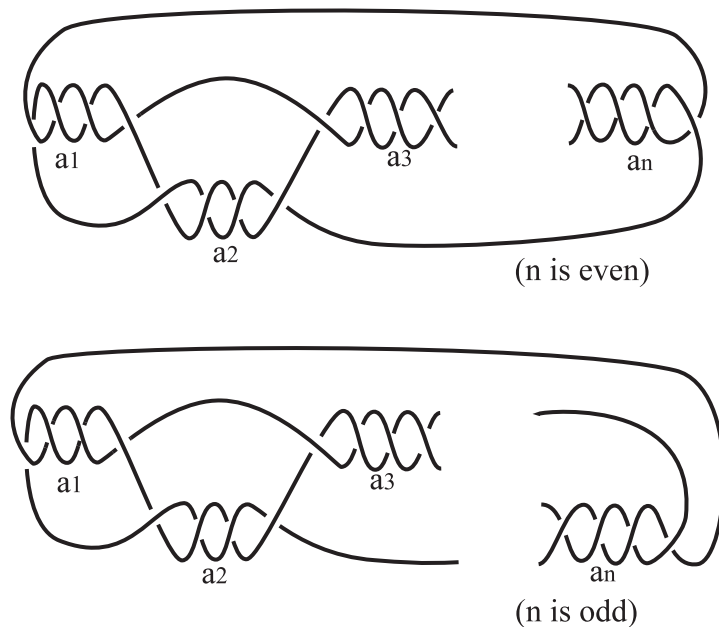


Figure 2.1: Two-bridge knots

There are several frequently studied families of knots, e.g. 2-bridge knots, torus knots and pretzel knots.

A 2-bridge knot is a knot which meets a projection plane  $\mathbf{R}^2$  in four points and intersects each of the two half spaces defined by  $\mathbf{R}^2$  in two arcs. These knots have been classified by Schubert [35] in 1956. A 2-bridge knot can be represented by a pair of coprime numbers in Schubert's notation  $S(p, q)$  where  $p > 0$ ,  $-p < q < p$  and  $q$  is an odd number. Any 2-bridge knot has a 4-plat representation, as shown as Figure 2.1, where  $a_i$  indicates  $|a_i|$  crossing points with proper signs. This regular projection can be denoted by  $C(a_1, a_2, \dots, a_n)$ , known as *Conway's normal form*.

A torus knot is a knot that can be embedded in the unknotted standard torus  $T$  in  $S^3$ . A  $(p, q)$ -torus knot is the knot contained in  $T$  that represents homologically  $p$  longitudes and  $q$  meridians of the unknot. Such a simple

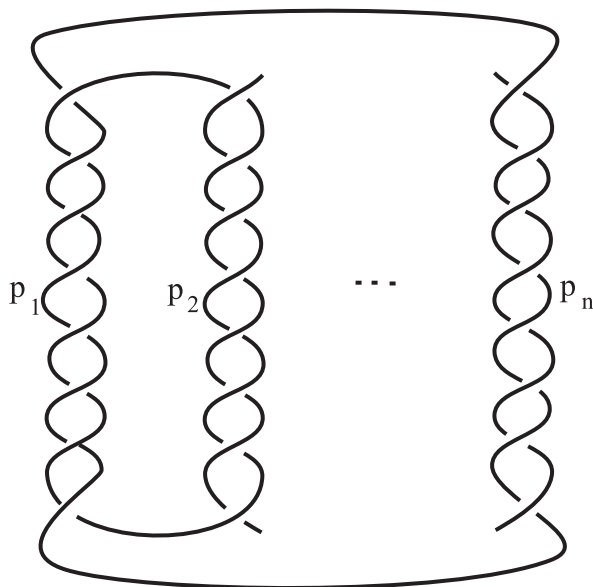


Figure 2.2: Pretzel knot

closed curve exists if and only if  $p$  and  $q$  are coprime.

For non-zero integers  $p_1, p_2, \dots, p_n$ , the link with the regular projection shown in Figure 2.2 is called the pretzel link and denoted by  $P(p_1, p_2, \dots, p_n)$ , where  $p_i$  indicates  $|p_i|$  crossings with proper signs.

## 2.1 Signature defined by using Seifert matrix and Seifert surface

The definition of the signature of knots and links was developed by Trotter [40] and Murasugi [28]. See the textbooks, for example [33] or [20].

Let  $L$  be an oriented link embedded in  $S^3$ . A *Seifert surface* for an oriented link  $L$  in the 3-sphere  $S^3$  is a connected, orientable, compact 2-manifold  $F$  in  $S^3$  with the boundary the link  $L$ . It has been shown by

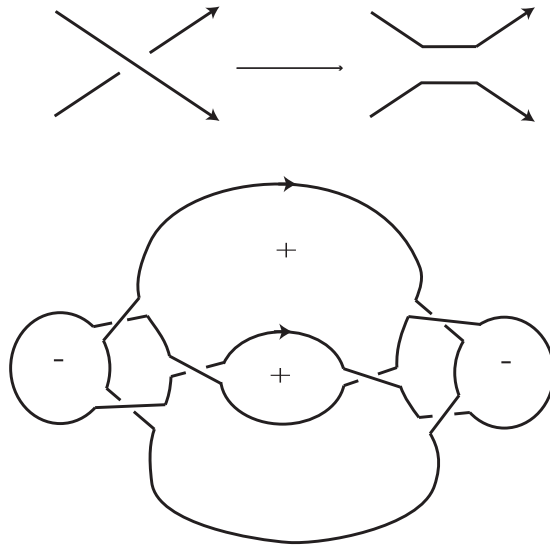


Figure 2.3: Seifert's algorithm

Seifert [37] that there always exists such an orientable surface bounding a knot. He gave an algorithm, known as Seifert algorithm, to construct such surfaces. The algorithm is as follows. Given an oriented knot diagram, smooth each crossing according to the orientation to get a collection of circles, known as Seifert circuits (or Seifert circles) which span mutually disjoint disks. Join these disks together by half-twisted bands at the crossings, then an orientable surface bounding the knot will be constructed. See Figure 2.3 for an illustration.

Associated to the Seifert surface  $F$  of genus  $g$  for  $L$  is the *Seifert pairing*  $f : H_1(F; \mathbf{Z}) \times H_1(F; \mathbf{Z}) \longrightarrow \mathbf{Z}$  defined by  $f(x, y) = \text{lk}(x, y^+)$ , where  $x, y \in H_1(F; \mathbf{Z})$  and  $y^+$  is the cycle carried by  $y \times \{1\}$  in the bicollar  $F \times [0, 1]$  with  $F$  identifies with  $F \times \{0\}$ . For an oriented knot, we can choose a basis  $e_1, e_2, \dots$ , and  $e_{2g}$  for  $H_1(F; \mathbf{Z})$  as a  $\mathbf{Z}$ -module. Then, define the associated *Seifert matrix*  $V = (v_{i,j})$  to be the  $2g \times 2g$  integral matrix with entries

$v_{i,j} = \text{lk}(e_i, e_j^+)$ . For a genus  $g$  Seifert surface bounding a 2-component link, the associated Seifert matrix should be a  $(2g + 1) \times (2g + 1)$  integral matrix with entries defined similarly.

For an oriented link in  $S^3$  with Seifert matrix  $V$ , one can define the *signature* of  $L$  to be  $\sigma(L) = \sigma(V + V^T)$  where  $\sigma(M)$  is the difference between the number of the positive eigenvalues and the number of negative eigenvalues of a symmetric matrix  $M$ . Note that the signature of a link is up to the relative orientation for each component. The signature is an invariant for oriented links.

## 2.2 Alternative way to calculate the signature for a link

### 2.2.1 Definition of Goeritz matrix

A *Goeritz matrix* for a link, not necessarily oriented, is a matrix of integers constructed in the following way [9]: Let  $D$  be a connected diagram of a link  $L$  and let the regions of the diagram be colored black and white in checkerboard way, leaving the region denoted by  $S_0$  at infinity white. Given this coloring, one can assign an incidence number  $\zeta(c)$  to each crossing  $c$ , as in Figure 2.4. Let  $S_0, S_1, \dots, S_n$  be the white regions of the diagram. Define a “pre-Goeritz matrix” to be the  $(n + 1) \times (n + 1)$  matrix having the terms  $\{g_{ij}\}$  given for  $i \neq j$  by

$$g_{ij} = \sum \zeta(c),$$

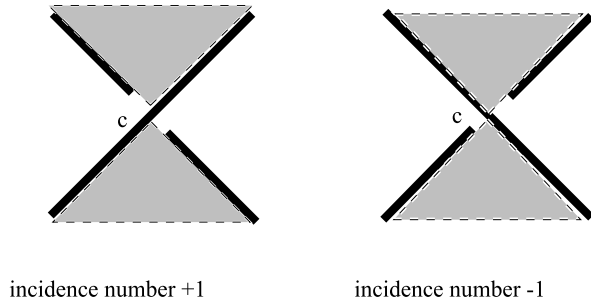


Figure 2.4: Incidence number

where the sum is taken over all the crossings at which  $S_i$  and  $S_j$  are connected. Define the diagonal terms by

$$g_{ii} = - \sum_{j \neq i} g_{ij}.$$

The related *Goeritz matrix* is the matrix with the row and the column indexed by zero deleted out of this one. Thus  $G$  is an integral symmetric  $n \times n$  matrix  $\{g_{ij} : 1 \leq i, j \leq n\}$ . Note that  $G$  depends on the diagram chosen for the link  $L$ , and also on the labeling of those white regions.

## 2.2.2 Alternative definition for Goeritz matrix

For an unoriented surface  $F$  bounding a link  $L$ , Gordon and Litherland [12] gave an alternative definition for a Goeritz matrix. Let  $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$  be a generator system for  $H_1(F; \mathbf{Z})$ . Then the  $(i, j)$ -entry of the Goeritz matrix  $G_F(L)$  is defined as  $\text{lk}(\alpha_i, \tau\alpha_j)$ , where  $\tau\alpha_j$  is the orientation double cover of  $\alpha_j$ .

With this definition of the Goeritz matrix, for an orientable Seifert surface with Seifert matrix  $V$ , the relation between the Goeritz matrix  $G$  for this

surface and  $V$  is that  $G = V + V^T$ .

### 2.2.3 Signature of a link by Gordon and Litherland

Then as follows, Gordon and Litherland [12] has shown how to define a quadratic form related with Goeritz matrix by using any spanning surface, and related the signature of this form to the signature of a link.

**Lemma 2.2.1 ([12])** *Let  $F$  be any surface bounding an unoriented link  $L$ , and let  $\bar{L}$  denote the link  $L$  together with some orientation on each component of it. Then the signature  $\sigma(\bar{L})$  can be calculated out of the Goeritz matrix  $G_F$  and the modified normal Euler number  $\bar{e}(F)$ , namely we have*

$$\sigma(\bar{L}) = \sigma(G_F) + \frac{1}{2}\bar{e}(F). \quad (2.1)$$

**Definition 2.2.2 ([12])** *Let  $L = K_1 \cup K_2 \cup \cdots \cup K_m$  be an  $m$ -component link with a connected surface  $F$  bounding it. The normal Euler number  $e(F)$  of it is defined to be*

$$- \sum_{i=1}^m \text{lk}(K_i, K'_i),$$

where  $K'_i$  is the intersection of  $F$  and the boundary of the regular neighborhood of  $K_i$  in  $S^3$ , i.e.  $K'_i = F \cap \partial N(K_i)$ , with orientation parallel to that of  $K_i$ .

We also define  $\bar{e}(F)$  to be

$$- \sum_{i,j=1}^m \text{lk}(K_i, K'_j).$$

It is clear that  $e(F) = \bar{e}(F) + 2\Lambda(L)$  where  $\Lambda(L)$  denotes the total linking number of the link  $L$ , namely,  $\Lambda(L) = \sum_{i < j} \text{lk}(K_i, K_j)$ . We call  $\bar{e}(F)$  the *modified normal Euler number*.

In fact, Gordon and Litherland [12] proved that  $\sigma(G_F) + \frac{1}{2}e(F)$  is an unoriented link invariant. Given two unoriented surfaces  $S_1$  and  $S_2$  properly embedded in  $D^4$ , bounding the link  $L$  in  $S^3 = \partial D^4$ , then consider the double branched cover  $X(S_1) \cup_{\partial} (-X(S_2))$  of  $S^4 = D^4 \cup_{\partial} D^4$  along  $S_1 \cup_{\partial} (-S_2)$ . By Novikov additivity, we have

$$\text{sign}(X(S_1) \cup_{\partial} (-X(S_2))) = \text{sign}(X(S_1)) - \text{sign}(X(S_2)),$$

and on the other hand by the G-signature theorem [11], we have

$$\begin{aligned} \text{sign}(X(S_1) \cup_{\partial} (-X(S_2))) &= 2 \text{sign}(S^4) - \frac{1}{2}e(S_1 \cup_{\partial} (-S_2)) \\ &= -\frac{1}{2}e(S_1 \cup_{\partial} (-S_2)) \\ &= -\frac{1}{2}e(S_1) + \frac{1}{2}e(-S_2). \end{aligned}$$

Then we have  $\text{sign}(X(S_1)) + \frac{1}{2}e(S_1) = \text{sign}(X(S_2)) + \frac{1}{2}e(S_2)$ , namely, it is an unoriented link invariant.

Note that the signature of a smooth compact 4-manifold  $X$ , is the number of the positive eigenvalues minus the number of the negative eigenvalues of its intersection form  $Q : H_2(X, \mathbf{Z})/\text{Tor} \otimes H_2(X, \mathbf{Z})/\text{Tor} \longrightarrow \mathbf{Z}$ .

## 2.3 Linking form of a link

### 2.3.1 Linking form of a 3-manifold

Let  $M$  denote a closed, oriented 3-dimensional manifold. Denote the torsion part of  $H_1(M)$  by  $\text{Tor } H_1(M)$ . A chain complex of  $M$  is as follows:

$$\dots \longrightarrow C_2(M) \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} 0,$$

where  $C_i(M)$  is the abelian group of  $i$ th simplex of  $M$ , and  $\partial_i$  is the boundary homomorphism ( $i = 1, 2, \dots$ ). Suppose that  $x, y \in \text{Tor } H_1(M)$  are represented by 1-cycles  $a$  and  $b$ . There exists  $n \in \mathbf{Z}$  such that  $[nb]$  is homologous to zero in  $H_1(M)$  and  $nb$  forms the boundary of some 2-chain, say  $\Delta$  in  $C_2(M)$ , i.e.  $\partial_2(\Delta) = nb$ . Define a bilinear form  $\lambda : \text{Tor } H_1(M; \mathbf{Z}) \times \text{Tor } H_1(M; \mathbf{Z}) \longrightarrow \mathbf{Q}/\mathbf{Z}$  as follows:

$$(x, y) \mapsto \text{Int}(a, \Delta)/n,$$

where  $\text{Int}$  denotes the intersection number between a 1-cycle and a 2-chain. This bilinear form  $\lambda$  is called the *linking form* on the 3-manifold  $M$ .

### 2.3.2 Relation between linking form and Goeritz matrix

Let  $D(L)$  denote the double branched cover of  $S^3$  over a link  $L$ . Let  $H_1(D(L); \mathbf{Z})$  be its first homology group over  $\mathbf{Z}$ . Then the bilinear form defined on  $D(L)$

$$\lambda : \text{Tor } H_1(D(L); \mathbf{Z}) \times \text{Tor } H_1(D(L); \mathbf{Z}) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

is called *the linking form for the link  $L$* . A Goeritz matrix  $G$  of the link  $L$  is a relation matrix for  $H_1(D(L); \mathbf{Z})$ , and the first homology group of  $D(L)$  and the linking form on  $D(L)$ ,  $(H_1(D(L); \mathbf{Z}), \lambda)$ , can be calculated out of the Goeritz matrix. Precisely speaking,  $H_1(D(L); \mathbf{Z}) = \mathbf{Z}^n / \text{Im}(G)$  where  $n$  is the size of the Goeritz matrix  $G$ , and the linking form is given by  $\pm G^{-1}$  if  $H_1(D(L); \mathbf{Z})$  is finite, i.e.  $\lambda(g_i, g_j) = \pm (G^{-1})_{ij} \pmod{1}$  for suitably chosen generators  $g_i$  and  $g_j$  of  $H_1(D(L); \mathbf{Z})$ . Here the sign  $\pm$  depends on the orientation.

**Remark 2.3.1** *It is known that the absolute value of the determinant of a Goeritz matrix of a link is the order of  $H_1(D(L); \mathbf{Z})$ , and hence an invariant of the link.  $\lambda$  is also a knot invariant.*

## 2.4 Integral binary quadratic form

We list some basic facts of number theory here, which will be applied in our later work.

**Definition 2.4.1** (see for example [18]) *The binary form with two variables  $x, y$*

$$f(x, y) = ax^2 + bxy + cy^2$$

*is called a binary quadratic form and the constants  $a, b$  and  $c$  are called its coefficients. If  $a, b, c \in \mathbf{Z}$ , then it is called an integral binary quadratic form.*

**Definition 2.4.2** (see for example [18]) *For the binary quadratic form*

$f(x, y) = ax^2 + bxy + cy^2$ , the value

$$D = b^2 - 4ac$$

is called the discriminant of the form  $f(x, y)$ .

A binary quadratic form  $f(x, y) = ax^2 + bxy + cy^2$  can be presented by a matrix in the following way:

$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Its discriminant is given by the matrix as follows:

$$D = -4 \begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix}.$$

Therefore, a matrix  $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$  with determinant  $d$  corresponds to a binary quadratic form  $f(x, y) = ax^2 + bxy + cy^2$  with discriminant  $D = -4d$  in a one-to-one way.

In this paper, we only consider integral binary quadratic forms.

**Definition 2.4.3** (see for example [18]) *The integral binary quadratic form  $f(x, y) = ax^2 + bxy + cy^2$  is equivalent to  $f'(x', y')$ , if there exists an integral*

unimodular matrix  $\begin{pmatrix} r & s \\ u & v \end{pmatrix}$  such that

$$\begin{aligned} f(x, y) &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} r & u \\ s & v \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}. \end{aligned}$$

Here  $rv - su = \pm 1$  and  $r, s, u, v \in \mathbf{Z}$ . Note that if  $f'(x', y')$  is given by the symmetric matrix  $\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}$ , then

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \begin{pmatrix} r & u \\ s & v \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} r & s \\ u & v \end{pmatrix}.$$

**Definition 2.4.4** ([18]) *The integral binary quadratic form  $f(x, y) = ax^2 + bxy + cy^2$  with  $D < 0$  is of reduced form if  $c > a \geq b > -a$  or  $c \geq b \geq 0$  with  $a > 0$ .*

**Lemma 2.4.5** ([18]) *Each equivalence class of integral binary quadratic forms with discriminant  $D < 0$  has one and only one representative of reduced form.*

**Definition 2.4.6** ([18]) *The integral binary quadratic form  $f(x, y) = ax^2 + bxy + cy^2$  with  $D > 0$  is of reduced form if  $a > 0, b < 0, c < 0, a + b + c < 0$  and  $a - b + c > 0$ .*

**Lemma 2.4.7** ([18]) *Each equivalence class of integral binary quadratic forms*

*with discriminant  $D > 0$  has one and only one representative of reduced form.*

Therefore, it is enough to compute only the integral binary quadratic forms of reduced forms, if we want to enumerate all the quadratic forms with some fixed discriminant.

# Chapter 3

## Various orientable genera of knots

In this chapter we give a survey of the known results about various orientable genera of knots.

### 3.1 Three-dimensional knot genus

The notion of the knot genus was first introduced by Seifert [37], and it holds an important position in knot theory. It is one of the most basic but important knot invariants. There is a general algorithm to determine the genus of a knot [36], but its application is complicated.

**Definition 3.1.1** *The genus of a link  $L$ , denoted by  $g(L)$ , is the least genus of all its Seifert surfaces.*

Note that it follows immediately that  $K$  is the unknot if and only if it has genus 0. Also if  $K$  has a Seifert surface of genus 1 and  $K$  is known not to be

the unknot, then  $g(K) = 1$ . Also from the Seifert's algorithm to construct a surface from a knot diagram  $D$ , we know that  $g(K) \leq 1/2(n - s - 1)$  if  $D$  has  $n$  crossings and  $s$  Seifert circles.

With some exceptions for special cases, for a long time the only known effective method to compute the genus of a knot  $K$  was to show the inequality

$$g(K) \geq 1/2 \text{ degree}(\Delta(K)),$$

proved by Seifert [37], is actually equality. Murasugi [26] in 1958 and Crowell [3] in 1959 independently showed the equality for alternating knots and then, the computation of the genera of these knots follows from the equality. They used rather combinatorial arguments to show that by applying Seifert's algorithm to an alternating projection of an oriented link one obtains a Seifert surface of minimal genus. In contrast with the algebraic approach, it was shown by Gabai [7] in 1984 using foliations that these surfaces are actually of the minimal genus.

**Theorem 3.1.2** ([26, 3, 7]) *Let  $K$  be an oriented knot in  $S^3$ . If  $S$  is a surface obtained by applying Seifert's algorithm to an alternating projection of  $K$ , then  $S$  is a Seifert surface of minimal genus. Also the minimal genus is one half of the degree of the Alexander polynomial of  $K$ .*

Unfortunately however, some of the simple knots, including some of the pretzel knots and all the Kinoshita-Terasaka knots, have vanishing Alexander polynomial and the genera of these knots were unknown. Actually the question of finding the genera of the pretzel knots dates back to 1934 when

Seifert [37] discovered the above inequality, showed that the pretzel knots were nontrivial and Alexander polynomial vanished for some of them. In 1986 Gabai [8] provided a way to construct the minimal genus Seifert surface for these knots by a geometrical approach using foliation and sutured manifold, and hence with practice the methods the genera of these knots could be computed.

Then in 2004 Ozsváth and Szabó [31] completely determined the knot genus of any given knot by applying their knot Floer homology.

**Theorem 3.1.3 ([31])** *For any knot  $K$  embedded in the 3-sphere  $S^3$ , let*

$$\deg H_{i,j}(K) = \max\{i \in \mathbf{Z} \mid \oplus_j H_{i,j}(K) \neq 0\}$$

*denote the degree of the knot Floer homology. Then  $g(K) = \deg H_{i,j}(K)$ . In particular knot Floer homology distinguishes every non-trivial knot from the unknot.*

It has already been noted that though it is easy to define some numerical knot invariants by minimizing some geometric phenomenon associated with it, such invariants are hard to calculate and difficult to use. The knot genus however, has a utility that arises from the additivity of itself [34].

**Theorem 3.1.4** *For any two knots  $K_1$  and  $K_2$ ,*

$$g(K_1 + K_2) = g(K_1) + g(K_2).$$

**Corollary 3.1.5** *A knot of genus 1 is prime.*

**Corollary 3.1.6** *A knot can be expressed as a finite sum of prime knots.*

## 3.2 Four-dimensional knot genus

If we consider the surfaces bounding the knot embedded in  $B^4$  instead of  $S^3$ , we can define the 4-dimensional genus, also called slice genus or 4-ball genus.

**Definition 3.2.1** *The 4-dimensional genus  $g^*(K)$  of a knot  $K$  is the minimal genus of the connected, orientable surface embedded in the 4-ball  $B^4$  bounding the knot  $K$ .*

This invariant has been studied extensively. It is known to be bounded from below by half the classical signature and the Tristram-Levine signature [16, 17, 27, 39].

Also it is not difficult to see that  $g^*(K)$  is a lower bound for the unknotting number  $u(K)$ . By definition,  $u(K)$  is the minimum number of crossing changes (from "over" to "under" or vice versa) needed to change  $K$  to the unknot, where the minimum is taken over all the possible knot diagrams of  $K$  and all possible sets of crossing changes.

Milnor gave a conjecture regarding the 4-dimensional knot genus of a torus knot stated as follows.

**Conjecture 3.2.2** *(Milnor's conjecture.) The four-dimensional genus of the type  $(p, q)$ -torus knot is  $(p - 1)(q - 1)/2$ .*

This conjecture was first to be proved true by Kronherimer and Mrowka in 1994 by using gauge-theoretic methods, as it is easy to show that this

number  $(p-1)(q-1)/2$  of crossing changes will undo the torus knot, and  $(p-1)(q-1)/2$  is the unknotting number.

Then in 2004, with the discovery of Khovanov homology, Rasmussen in [32] gave another combinatorial proof by using Khovanov homology, which has been considered as a surprising application of Khovanov homological theory.

### 3.3 Concordance genus and relations with others

Artin in 1926 [22] constructed certain knotted 2-spheres in  $R^4$ . The intersection of these surface knots with the 3-space  $R^3$  is a nontrivial knot in  $R^3$ . Thus, to identify which knots can appear as such slices of knotted surface became a natural problem. Such nontrivial knots are called *slice knots*. It was proved that some knots are not slice by Murasugi [27] and Fox and Milnor [6] in 1960s. The details follow in later chapters.

**Definition 3.3.1** *Two knots  $K_0$  and  $K_1$  in  $S^3$  are called concordant if there is a smooth embedding of the annulus  $S^1 \times [0, 1]$  into  $S^3 \times [0, 1]$  having boundary the knot  $K_0$  and  $-K_1$  in  $S^3 \times \{0\}$  and  $S^3 \times \{1\}$  respectively.*

Knot concordance is an equivalence relation, and the set of equivalence classes forms a countable abelian group under the operation induced by knot connected sum. The concordance group was introduced by Fox and Milnor [6] in 1966. A knot, called a *slice knot*, which bounds an embedded disk in

$B^4$  represents the trivial element in this group. In other words, a slice knot is concordant to the unknot.

In 1977 Gordon [10] made a conjecture that any knot that bounds a surface of genus  $g$  in  $B^4$  is concordant to a knot of 3-dimensional genus  $g$ . This initiates the study of genus in the concordance category.

Nakanishi [29] and Casson observed that there are knots that bound surfaces of genus 1 in  $B^4$  but which are not concordant to the knots of 3-dimensional genus 1, which contradicts the above conjecture. Nakanishi [29] actually proved the following results.

**Theorem 3.3.2** *For any positive integer  $n$ , there is a prime knot  $K$  such that  $n = g^*(K) < g(K_0)$  for any  $K_0$  concordant to  $K$ .*

This was the starting point of the definition of concordance genus by Livingston [21].

**Definition 3.3.3** ([21]) *The concordance genus of a knot  $K$ , denoted by  $g_c(K)$ , is the minimum genus among all the knots concordant to  $K$ .*

It is shown by Livingston [21] that this invariant can be arbitrarily large even for knots of 4-dimensional genus 1. For prime knots of 10 or fewer crossings, the concordance genus have been claimed to be resolved by using invariants of Levine's algebraic concordance group, and twisted Alexander polynomials (Casson-Gordon invariants).

# Chapter 4

## Crosscap numbers of knots

### 4.1 Three-dimensional crosscap number

The crosscap number  $\gamma(K)$  of a knot  $K$  was first introduced by Clark in 1978.

Let  $S$  be a connected, non-orientable surface which bounds a knot  $K$ . It is the boundary connected sum of some Möbius bands. Define  $\gamma(S)$  to be the first Betti number of  $S$ .

**Definition 4.1.1** ([2]) *Let  $S$  be any non-orientable surface bounding  $K$  in  $S^3$ . Define the crosscap number for the knot  $K$  in the following way:*

$$\gamma(K) := \min\{\gamma(S) \mid \partial S = K, S \text{ is non-orientable surface}\}.$$

**Remark 4.1.2** *Note that for the unknot  $U$ , we define the crosscap number  $\gamma(U) := 1$ .*

For completeness, Clark defined the crosscap number for the unknot to

be 0 instead of 1, and at the same time, he gave a necessary and sufficient condition for the crosscap number 1 knots.

**Proposition 4.1.3** ([2]) *A knot  $K$  has crosscap number 1 if and only if it is a  $(2, n)$ -cable knot.*

**Definition 4.1.4** (see for example [20]) *A  $(p, q)$ -cable knot about some knot  $K$  is a satellite knot which runs  $p$  times around the meridian of  $K$  and at the same time runs  $q$  times around the longitude of  $K$ .*

The common process of constructing a satellite knot is as follows. Start with a knot  $K$  in a solid torus  $T$ . Let  $f : T \rightarrow S^3$  be an embedding such that  $f(T)$  is a regular neighbourhood of another knot  $K_0$  in  $S^3$ . Then  $f(K)$  is called *satellite* of  $K_0$ . If we take  $K$  to be a  $(2, n)$ -torus knot, and then  $f(K)$  is called a  $(2, n)$ -cable knot.

He also proved the following inequality and asked whether there exist some knots for which the equality holds.

$$\gamma(K) \leq 2g(K) + 1. \tag{4.1}$$

Until 1995, Murakami and Yasuhara [24] brought forward the first example for which the above equality holds by showing that  $\gamma(7_4) = 3$ .

For the behavior of the crosscap number of a knot under the connected sum, Clark gives the following inequality:

**Proposition 4.1.5** ([2]) *Let  $K_1$  and  $K_2$  be the two knots. We have*

$$\gamma(K_1) + \gamma(K_2) - 1 \leq \gamma(K_1 \# K_2) \leq \gamma(K_1) + \gamma(K_2).$$

Murakami and Yasuhara [24] gave a necessary and sufficient condition for the equalities, which is stated as:

**Proposition 4.1.6 ([24])** *Let  $K_1$  and  $K_2$  denote any two knots, then we have  $\gamma(K_1 \sharp K_2) = \gamma(K_1) + \gamma(K_2)$  if and only if  $\gamma(K_1) = \Gamma(K_1)$  and  $\gamma(K_2) = \Gamma(K_2)$ , where  $\Gamma(K)$  denotes the minimum first Betti number of unoriented surfaces, including orientable and non-orientable ones, bounding the knot  $K$ .*

In general, it is hard to determine the crosscap number for a knot. But for the class of torus knots, Teragaito [38] determines the crosscap numbers. The main theorem in [38] is stated as follows:

**Theorem 4.1.7 ([38])** *Let  $K$  be the torus knot  $T(p, q)$  ( $p, q > 0$ ). Then  $\gamma(K) = N(p, q)$  if  $pq$  is even, and  $\gamma(K) = N(pq - 1, p^2)$  if  $pq$  is odd. Here  $N(p, q)$  is the minimum genus of closed, connected, non-orientable surfaces contained in the lens space  $L(p, q)$ .*

According to [38], the crosscap numbers of torus knots are additive under connected sum, namely, we have

$$\gamma(K_1 \sharp K_2 \sharp \cdots \sharp K_n) = \gamma(K_1) + \gamma(K_2) + \cdots + \gamma(K_n),$$

where  $K_i$  ( $i = 1, 2, \dots, n$ ) is a torus knot.

In April 2005, Hirasawa and Teragaito [14] presented an algorithm to determine the crosscap numbers for all the 2-bridge knots. Here in this thesis, a two-bridge knot  $S(p, q)$  is presented by a pair of coprime integers  $(p, q)$  where  $p$  is odd. Given a fraction  $p/q$ , consider a continued fraction

expansion,  $p/q = r + [b_1, b_2, \dots, b_n]$ , of it. Namely,

$$p/q = r + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{\dots + \frac{1}{b_n}}}}.$$

In fact, an algorithm to obtain a shortest continued fraction expansion is presented in [14], and the crosscap number of the 2-bridge knot  $S(p, q)$  can be determined in terms of the length  $n$  of this shortest continued fraction expansion.

**Theorem 4.1.8** ([14]) *Let  $K$  be a 2-bridge knot denoted by  $S(p, q)$ . If a shortest expansion of  $p/q$  contains an odd coefficient or  $\pm 2$ , then we have  $\gamma(K) = n$ . Otherwise, we have  $\gamma(K) = n + 1$ .*

Note that in [14] the authors also gave an infinite series of 2-bridge knots with  $\gamma(K) = 2g(K) + 1$ .

Let  $K = P(p_1, p_2, \dots, p_n)$  be a non-trivial pretzel knot. In order to make it a knot, either of the following conditions must be satisfied:

- (a) one and only one among  $p_1, p_2, \dots, p_n$  is even and other are odd;
- (b)  $n$  is odd and all of  $p_1, p_2, \dots, p_n$  are odd.

A result by Ichihara and Mizushima is as follows.

**Theorem 4.1.9** ([15]) *Let  $K = P(p_1, p_2, \dots, p_n)$  be a non-trivial pretzel knot. When the condition (a) is satisfied we have  $\gamma(K) = n - 1$ , and otherwise when the condition (b) satisfied we have  $\gamma(K) = n$ .*

## 4.2 Four-dimensional crosscap number

In [25] Murakami and Yasuhara defined the 4-dimensional version of the crosscap number.

**Definition 4.2.1** *The 4-dimensional crosscap number, or the 4-dimensional non-orientable genus,  $\gamma^*(K)$  is the minimum first Betti number of non-orientable surfaces embedded in  $B^4$  bounding the knot  $K$ .*

**Remark 4.2.2** *The 4-dimensional crosscap number of a slice knot is defined to be 0 instead of 1.*

From the definitions it is not hard to see  $\gamma^*(K) \leq \gamma(K)$ . Same as the 3-dimensional crosscap number  $\gamma(K)$ , there is the following relation between the 4-dimensional crosscap number and 4-dimensional genus.

**Proposition 4.2.3** ([25])

$$\gamma^*(K) \leq 2g^*(K) + 1.$$

In [24] the authors have proved that the knot  $7_4$  satisfies the equality  $\gamma(K) = 2g(K) + 1$  ( Note that more examples for which the equality  $\gamma(K) = 2g(K) + 1$  holds are given by Hirasawa and Teragaito in [14]). Therefore Murakami and Yasuhara made the following conjecture.

**Conjecture 4.2.4** ([25]) *There exists a knot  $K$  such that the equality  $\gamma^*(K) = 2g^*(K) + 1$  holds.*

Here we are going to discuss some basic property of knots with 4-dimensional crosscap number 1. If a knot can be unknotted by adding a non-orientable band as illustrated in Figure 6.5, then it bounds a (possibly twisted) Möbius band in  $B^4$ .

Lickorish in [19] gave a necessary algebraic condition for this geometric phenomenon of knots.

**Theorem 4.2.5 ([19])** *Let  $D_K$  be the double cover of  $S^3$  branched along a knot  $K$ , which can be unknotted by adding a non-orientable band. Let*

$$\lambda : H_1(D_K) \times H_1(D_K) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

*be the linking form. Then  $H_1(D_K)$  is cyclic of order  $|\det K|$  and has a generator  $g$  such that  $\lambda(g, g) = \pm 1/\det K$ .*

Even if a knot  $K$  bounds a (twisted) Möbius band in  $B^4$ , it can not necessarily be created by adding a non-orientable band to the unknot  $U$ . See [24] for more details.

# Chapter 5

## Crosscap numbers of two-component links

Throughout this chapter we work in the smooth category, and knots and links we work with are embedded in the 3-sphere  $S^3$ .

The crosscap number of a knot  $K$  was introduced by Clark [2] in 1978. It is defined to be the minimum of the first Betti numbers of non-orientable surfaces bounding  $K$ . Various notations for the crosscap number of a knot have been used in the past research on it, see for example [2, 24, 38]. In the paper [42] we denote it by  $\gamma(K)$ .

Clark observed in [2] the inequality  $\gamma(K) \leq 2g(K) + 1$  and raised the question whether some knots exist for which the equality holds. Murakami and Yasuhara [24] brought forward a concrete calculation for the knot  $7_4$  which is the first example known to satisfy the equality above. It has been shown [14] that there exist numerous knots for which the equality holds.

Given a knot, generally it is hard to determine its crosscap number. Clark

gave a necessary and sufficient condition for the crosscap number 1 knots, which says that a knot has crosscap number 1 if and only if it is a  $(2, n)$ -cable knot. Recently the crosscap numbers for several families of knots, such as the torus knots in [38], the 2-bridge knots in [14], and the pretzel knots in [15], have been determined.

In this chapter, we define the crosscap number for two-component links and discuss some basic properties of it. By following the technique used in [24], we calculate the crosscap number of the two-component link  $6_3^2$  as an example. Here we use the notation of Rolfsen [33] to denote a link in his link table.

## 5.1 Definitions

The crosscap number of a knot was first introduced by Clark [2] in 1978.

**Definition 5.1.1** ([2]) *The crosscap number  $\gamma(K)$  of a knot  $K$  is the minimal number of the first Betti numbers of all the connected, non-orientable surfaces bounding  $K$ . The crosscap number of the unknot  $U$  is defined to be 1.*

Note that we define the crosscap number of the unknot to be 1, instead of 0 defined by Clark [2].

Clark also gave an upper bound for the crosscap number of a knot in terms of its genus.

**Proposition 5.1.2** ([2]) *Let  $K$  be a knot, and  $g(K)$  denote the genus of  $K$ .*

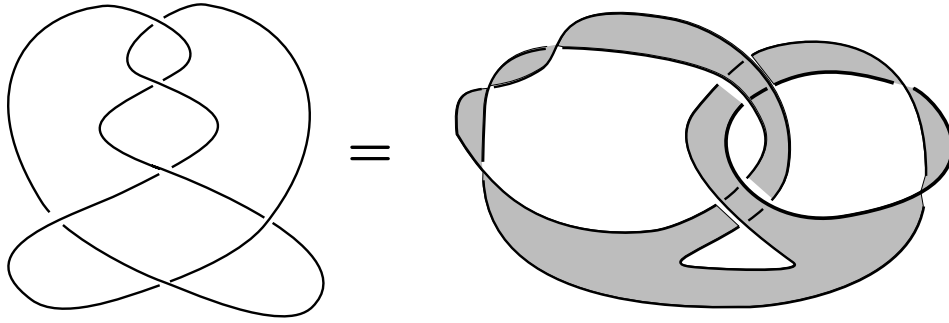


Figure 5.1: Non-orientable surface bounding  $6_2^2$  with  $\beta_1 = 2$

Then

$$\gamma(K) \leq 2g(K) + 1. \quad (5.1)$$

Beginning with the knot  $7_4$  proved by Murakami and Yasuhara in [24], it has been shown in [14] that numerous knots are suited for the equality in (5.1).

The crosscap number of a two-component link is defined similarly to that of a knot.

**Definition 5.1.3** *The crosscap number  $\gamma(L)$  of a two-component link  $L$  is the minimum of the first Betti numbers of connected, non-orientable surfaces bounding the link, i.e. we have*

$$\gamma(L) := \min\{\beta_1(F) \mid F \text{ is a connected non-orientable surface bounding } L\}.$$

It is not hard to see that for a 2-component link  $L$ , its crosscap number is at least 2, i.e.  $\gamma(L) \geq 2$ . This is because the projective plane  $\mathbf{R}P^2$  is a

closed surface with minimum first Betti number, and the first Betti number of the surface obtained by cutting two disks off  $\mathbf{R}P^2$  is 2.

Let  $6_2^2$  be the two-component link illustrated in Figure 5.1.

**Example 5.1.4** *We have  $\gamma(6_2^2) = 2$  as shown in the picture.*

## 5.2 Behavior of crosscap numbers under split union

In this section we will demonstrate how the crosscap number of a split union of two knots can be evaluated by the crosscap numbers of the knots.

**Definition 5.2.1** (See [20]) *A link  $L = L_1 \cup L_2$  is called splittable if there exists some 2-sphere  $S^2$  embedded in  $S^3$  bounding two 3-balls  $B_1$  and  $B_2$  with  $B_1 \cup B_2 = S^3$  such that  $L_i$  lies in the interior of  $B_i$  ( $i = 1, 2$ ). A split union of two knots  $K_1$  and  $K_2$ , denoted by  $K_1 \circ K_2$ , is the splittable link with the 2-sphere bounding two 3-balls such that  $K_i$  lies in the interior of  $B_i$  ( $i = 1, 2$ ).*

Firstly, we consider the crosscap number of the split union of any two knots. Actually, for the split union of any two knots, its crosscap number can be known in terms of genera and crosscap numbers of the knots. Thus, we have the following equality.

**Theorem 5.2.2** *Let  $K_1$  and  $K_2$  be any two knots. We have*

$$\begin{aligned} \gamma(K_1 \circ K_2) = \min\{\gamma(K_1) + 2g(K_2) + 1, \\ \gamma(K_2) + 2g(K_1) + 1, \gamma(K_1) + \gamma(K_2) + 1\}. \end{aligned}$$

*Proof:* Let  $G'$  be a connected, non-orientable surface which gives the minimum first Betti number for the split union  $K_1 \circ K_2$ . There exists a 2-sphere  $S$  separating  $K_1$  and  $K_2$  with  $S \cap G' \neq \emptyset$ . Take an innermost circle  $\alpha$  of  $S \cap G'$  in  $S$  without bounding any disk in  $G'$ . (We can always do this because we can remove the circle  $\alpha$  without changing the first Betti number by doing surgery on the surface  $G'$  if it bounds a disk in  $G'$ .)

If  $\alpha$  is a non-separating curve in  $G'$ , then we do surgery along  $\alpha$  on the surface  $G'$ , from which a new surface with smaller first Betti number, still bounding  $K_1 \circ K_2$ , connected, will appear. This contradicts the assumption that the surface  $G'$  realizes the crosscap number for  $K_1 \circ K_2$ .

Hence we may assume that  $\alpha$  is a separating curve which separates the surface  $G'$  into two surfaces, say  $G'_1$  and  $G'_2$ . Each of them bounds  $K_1$  or  $K_2$  separately; otherwise we can get a connected, non-orientable surface bounding  $K_1 \circ K_2$  with smaller first Betti number by doing surgery along  $\alpha$  on  $G'$ , which contradicts the minimality of the crosscap number. Then at least one of the two surfaces  $G'_1$  and  $G'_2$  is non-orientable. In all there are three possibilities:  $G'_1$  orientable and  $G'_2$  non-orientable,  $G'_1$  non-orientable and  $G'_2$  orientable, or both non-orientable. Therefore we get the inequality  $\min\{\gamma(K_1)+2g(K_2)+1, \gamma(K_2)+2g(K_1)+1, \gamma(K_1)+\gamma(K_2)+1\} \leq \gamma(K_1 \circ K_2)$ .

On the other hand, we may assume that the knot  $K_i$  bounds an orientable surface  $S_i$  with genus  $g(K_i)$  and a non-orientable surface  $G_i$  with first Betti number  $\gamma(K_i)$  ( $i = 1, 2$ ). Then, three non-orientable surfaces bounding  $K_1 \circ K_2$  will be produced if we connect the surfaces  $S_1$  and  $G_2$ ,  $G_1$  and  $S_2$ , and  $G_1$  and  $G_2$  by tubes. The first Betti numbers of these three surfaces are

$\gamma(K_1) + 2g(K_2) + 1$ ,  $\gamma(K_2) + 2g(K_1) + 1$  and  $\gamma(K_1) + \gamma(K_2) + 1$  respectively. This gives us an upper bound of the crosscap number of the split union, i.e.  $\gamma(K_1 \circ K_2) \leq \min\{\gamma(K_1) + 2g(K_2) + 1, \gamma(K_2) + 2g(K_1) + 1, \gamma(K_1) + \gamma(K_2) + 1\}$ .

Thus, we have the whole proof of the equality.  $\square$

**Corollary 5.2.3** *Let  $K_1$  and  $K_2$  be any two knots. Then the following inequalities hold:*

$$\begin{aligned} \gamma(K_1 \circ K_2) &\leq \gamma(K_1) + \gamma(K_2) + 1, & (5.2) \\ \gamma(K_1 \circ K_2) &\leq \gamma(K_1) + 2g(K_2) + 1, \\ \gamma(K_1 \circ K_2) &\leq \gamma(K_2) + 2g(K_1) + 1. \end{aligned}$$

Using Clark's inequality (5.1), we have

**Corollary 5.2.4** *Let  $K_1$  and  $K_2$  be any two knots. Then  $\gamma(K_1 \circ K_2) = \gamma(K_1) + \gamma(K_2) + 1$  if and only if  $\gamma(K_i) < 2g(K_i) + 1$ ,  $i = 1$  and  $2$ .*

This corollary is equivalent to the following.

**Corollary 5.2.5**  *$\gamma(K_1 \circ K_2) = \gamma(K_1) + \gamma(K_2)$  if and only if  $\gamma(K_i) = 2g(K_i) + 1$ ,  $i = 1$  or  $2$ .*

Note that when  $\gamma(K_1) < 2g(K_1) + 1$  and  $\gamma(K_2) = 2g(K_2) + 1$ , then  $\gamma(K_1 \circ K_2) = \gamma(K_1) + 2g(K_2) + 1$ ; when  $\gamma(K_1) = 2g(K_1) + 1$  and  $\gamma(K_2) < 2g(K_2) + 1$ , then  $\gamma(K_1 \circ K_2) = \gamma(K_2) + 2g(K_1) + 1$ . In fact, the crosscap number of  $K_1 \circ K_2$  is exactly equal to  $\gamma(K_1) + \gamma(K_2)$  in both cases.

If we apply the argument above to the case when  $K_2 = U$ , we have the following corollary.

**Corollary 5.2.6** *Let  $U$  denote the unknot and  $K$  be any knot. Then we have*

$$\gamma(K \circ U) = \gamma(K) + 1.$$

Next, by applying the homology theory, we discuss a little more the examples for which the equality in (5.2) holds.

**Proposition 5.2.7** *Let  $D(L)$  denote the double branched cover of  $S^3$  branched along the two-component link  $L$ . Then the minimum number of generators for  $H_1(D(L); \mathbf{Z})$  has the crosscap number  $\gamma(L)$  as an upper bound.*

*Proof:* Let  $F$  denote a non-orientable surface which has the minimum first Betti number,  $\gamma(L)$ , bounding  $L$ . Then corresponding to this surface, there is a  $\gamma(L) \times \gamma(L)$  Goeritz matrix built in the way of Gordon and Litherland [12, §2]. Then this Goeritz matrix becomes a relation matrix for  $H_1(D(L); \mathbf{Z})$ , see Chapter 2, from which the result follows.  $\square$

It is known that  $g(3_1) = 1$  and  $\gamma(3_1) = 1$ . Thus  $\gamma(3_1) < 2g(3_1) + 1$ , and by Corollary 5.2.4 we have  $\gamma(3_1 \circ 3_1) = 3 = \gamma(3_1) + \gamma(3_1) + 1$ . Now we prove this in terms of homology theory.

By connecting the two non-orientable surfaces bounding the trefoils with a tube, a non-orientable surface  $F$  bounding  $3_1 \circ 3_1$  can be built. see Figure 5.2. The Goeritz matrix corresponding to this surface with indicated generators  $\{a, b, c\}$  is as follows:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

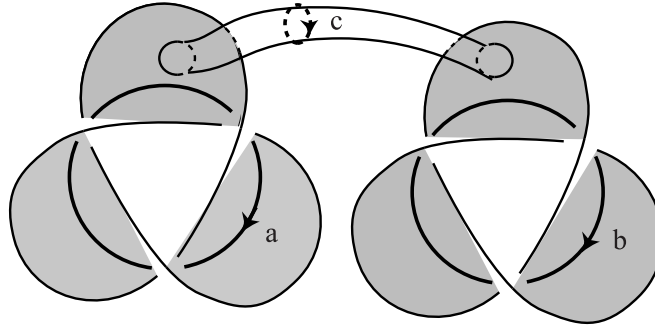


Figure 5.2: Tubular connection

The first homology group of the double cover of  $S^3$  branched over  $3_1 \circ 3_1$ , can be known as  $H_1(D(3_1 \circ 3_1); \mathbf{Z}) = \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}$ . By the fundamental theorem of abelian groups,  $H_1(D(3_1 \circ 3_1); \mathbf{Z})$  cannot be presented by a  $2 \times 2$  matrix.

So the crosscap number of the split union of two trefoil knots cannot be 2 by Proposition 5.2.7. Then we have  $\gamma(3_1 \circ 3_1) = 3 = \gamma(3_1) + \gamma(3_1) + 1$ .

### 5.3 Upper bounds of crosscap numbers of two-component links

Denote by  $\vec{L}$  and  $\overleftarrow{L}$  the two different relative orientations for a 2-component link  $L$ . Let  $g(\vec{L})$  and  $g(\overleftarrow{L})$  denote the genera of  $L$  under these two different orientations.

**Theorem 5.3.1** *With the notations above, we have*

$$\gamma(L) \leq 2 \min(g(\vec{L}), g(\overleftarrow{L})) + 2. \quad (5.3)$$

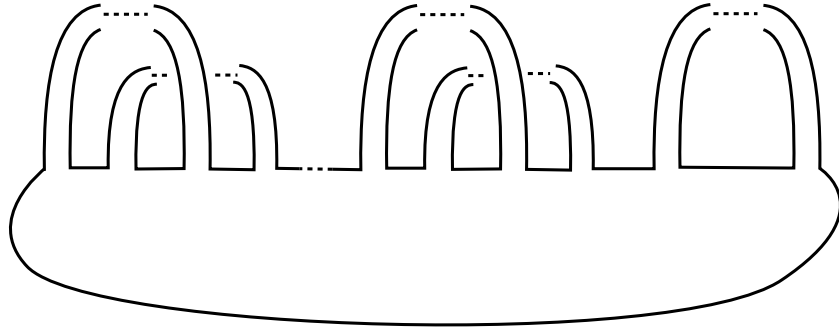


Figure 5.3: Standard form of a Seifert surface bounding a 2-component link

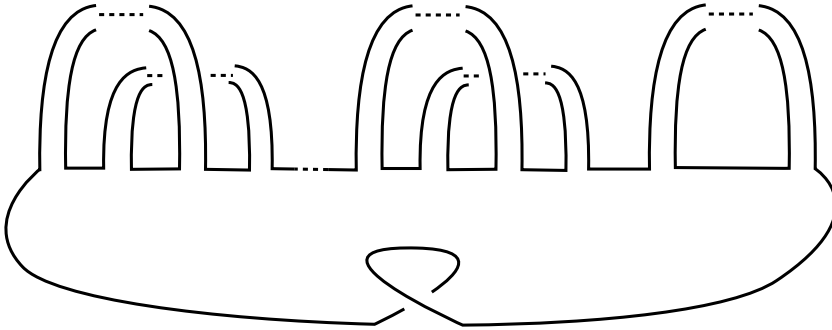


Figure 5.4: Surface after adding a half-twisted band

*Proof:* Denote  $\min(g(\vec{L}), g(\overleftarrow{L}))$  by  $g$ . Then there exists an orientable Seifert surface  $F$ , whose standard form is illustrated in Figure 5.3, with genus  $g$  bounding the 2-component link  $L$ , so the first Betti number of this surface becomes  $2g + 1$ . By adding a half twist to the surface, we obtain a non-orientable surface from  $F$  as illustrated in Figure 6.6, whose first Betti number is  $2g + 2$ . By the definition of the crosscap number, we know that the inequality holds.  $\square$

Does there exist any 2-component link for which the equality in (5. 3) holds?

There exists an infinite sequence of 2-component links  $T(2, 2n)$  ( $n \in \mathbf{Z}$ ) for which the equality in (3) holds, where  $T(p, q)$  denotes a torus knot or

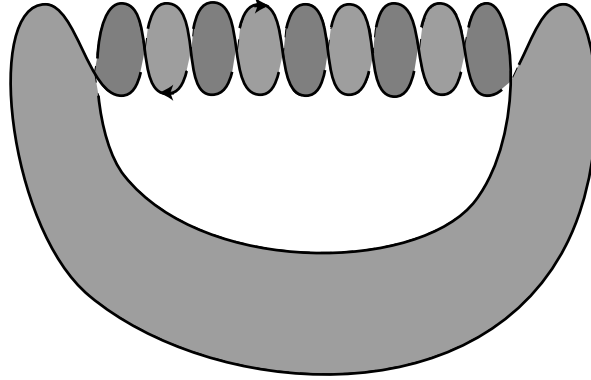


Figure 5.5: Torus link  $T(2, 10)$

link. Take an example of torus link  $T(2, 10)$  as illustrated in Figure 5.5. It bounds a genus 0 orientable Seifert surface, which gives us an upper bound 2 for the crosscap number of the link by using the inequality (5.3). Therefore we have the fact that the crosscap number of the torus link  $T(2, 2n)$  is 2, for which the equality in (3) holds.

Let  $n(L)$  denote the minimum crossing number of a link  $L$ . We can give another upper bound for the crosscap number of a two-component link in terms of  $n(L)$ . For a knot, Murakami and Yasuhara in [24] proved the following proposition.

**Proposition 5.3.2** ([24]) *Let  $n(K)$  denote the crossing number of a knot  $K$ . Then*

$$\gamma(K) \leq \frac{\lfloor n(K) \rfloor}{2}, \quad (5.4)$$

where  $\lfloor x \rfloor$  denotes the greatest integer that does not exceed  $x$ .

For a two-component link, we have the following result.

**Proposition 5.3.3** *Let  $L$  be a two-component link, excluding the unlink.*

Then, we have

$$\gamma(L) \leq \frac{\lfloor n(L) \rfloor}{2} + 1,$$

where  $\lfloor x \rfloor$  denotes the greatest integer that does not exceed  $x$ .

Note that for the Hopf link, the equality in Proposition 5.3.3 holds.

*Proof:* If  $L$  is a splittable link  $K_1 \circ K_2$ , then from Proposition 5.3.2 we have  $\gamma(K_i) \leq \frac{\lfloor n(K_i) \rfloor}{2}$  for  $i = 1, 2$ , and from Corollary 5.2.3 we have  $\gamma(L) \leq \gamma(K_1) + \gamma(K_2) + 1$ . Then the inequality  $\gamma(L) \leq \frac{\lfloor n(K_1) \rfloor}{2} + \frac{\lfloor n(K_2) \rfloor}{2} + 1$  follows. Hence the theorem holds for splittable links due to the fact that  $\frac{\lfloor n(K_1) \rfloor}{2} + \frac{\lfloor n(K_2) \rfloor}{2} \leq \frac{\lfloor n(L) \rfloor}{2}$ .

Now, let  $D$  be a link diagram of non-splittable link  $L$  with the minimum crossing number  $n(L)$ . Then, we have  $n(L) + 2$  regions of  $S^2$  divided by the link diagram. Color these regions black and white in a checkerboard way. Since  $L$  is a non-splittable link, all the regions of the same color can be connected to each other by half-twisted bands at the crossings. Both the white and black surfaces can be orientable, then  $\gamma(L)$  should be less than or equal to the first Betti number of these surfaces plus one, where the “1” indicates the added first Betti number by adding a half-twisted band in that case. Denote the numbers of black and white regions by  $n(b)$  and  $n(w)$  respectively. Note that the number of the edges is twice that of vertices. Then, by using a relation between Euler characteristic and the first Betti number together with Euler’s formula, we have

$$\gamma(L) \leq 2 + n(L) - \max\{n(b), n(w)\}.$$

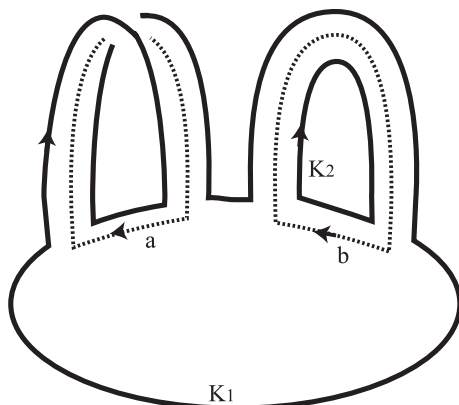


Figure 5.6: A crosscap number two surface

It is not hard to show that  $\max\{n(b), n(w)\} \geq \frac{1}{2}(n(L) + 2)$  in case that  $n(L)$  is even, and  $\max\{n(b), n(w)\} \geq \frac{1}{2}(n(L) + 3)$  in case that  $n(L)$  is odd. Therefore, the result follows.  $\square$

## 5.4 An example of calculation

In this section, we will calculate the crosscap number of the two-component link  $6_3^2$  as an example.

**Lemma 5.4.1** *Let  $L$  be a two-component link  $K_1 \cup K_2$ . Assume that it bounds a connected, non-orientable surface  $F$  with the first Betti number 2. Then we can choose a generator system for  $H_1(F; \mathbf{Z})$  such that the Goeritz matrix  $G_F(L)$  corresponding to this system is of the following form:*

$$G_F(L) = \begin{pmatrix} 2n + 1 & 2k \\ 2k & 2m \end{pmatrix}, \quad (5.5)$$

where  $k, m, n$  are integers.

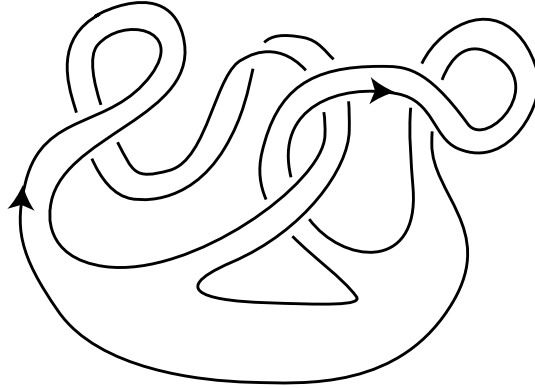


Figure 5.7: An example of a non-orientable surface with  $\beta_1 = 2$

*Proof:* We may assume that  $F$  is a disk with a non-orientable band and an orientable one as indicated in Figure 5.6, where each band may be knotted and linked with each other; see for example Figure 5.7. Choose a generator system  $\{a, b\}$  as in Figure 5.6 and orient the two components  $K_1$  and  $K_2$  so that the two boundaries of each band have the same orientations as that of the 1-cycle passing through it.

Then the Goeritz matrix of the surface  $F$ , refer to [12, §2], corresponding to this generator system is

$$\begin{pmatrix} \text{lk}(a, \tau a) & \text{lk}(a, \tau b) \\ \text{lk}(b, \tau a) & \text{lk}(b, \tau b) \end{pmatrix} = \begin{pmatrix} 2\delta(A, A) + 1 & 2\delta(A, B) \\ 2\delta(B, A) & 2\delta(B, B) \end{pmatrix},$$

where  $\tau(x)$  denotes the orientation double cover of a cycle  $x$ .

Here  $A$  and  $B$  are the bands which  $a$  and  $b$  pass through respectively, and  $\delta(X, Y)$  is the sum of signs of crossings of bands  $X$  and  $Y$  with signs determined as in Figure 5.8. Note that  $\text{lk}(a, \tau b) = \text{lk}(b, \tau a) = 2\text{lk}(a, b)$ , and that both of them are even. This finishes the proof.  $\square$

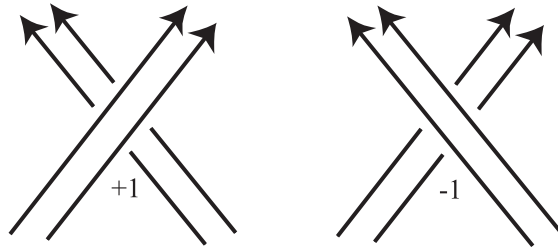


Figure 5.8: Crossing types of bands

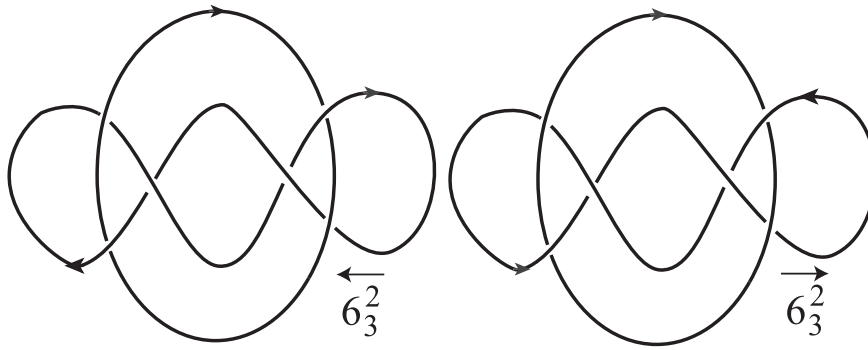


Figure 5.9: 2-component link  $6_3^2$  with two orientations

**Lemma 5.4.2** *With the orientations as above in the proof of Lemma 5.4.1, the linking number between  $K_1$  and  $K_2$  is  $m + 2k$  and the modified Euler number, see Chapter 2, of the surface  $F$  is  $(-2)$  times the sum of all the elements in the Goeritz matrix  $G_F(L)$ . Namely,*

$$\text{lk}(K_1, K_2) = m + 2k, \quad \bar{e}(F) = -2(2n + 1 + 2k + 2k + 2m).$$

*Proof:* According to the proof of the above theorem, we have  $m = \delta(B, B)$  and  $k = \text{lk}(a, b)$ . It is obvious that the calculation of the linking number between  $K_1$  and  $K_2$  includes these two parts, which gives us  $\text{lk}(K_1, K_2) = \delta(B, B) + 2\text{lk}(a, b) = m + 2k$ .

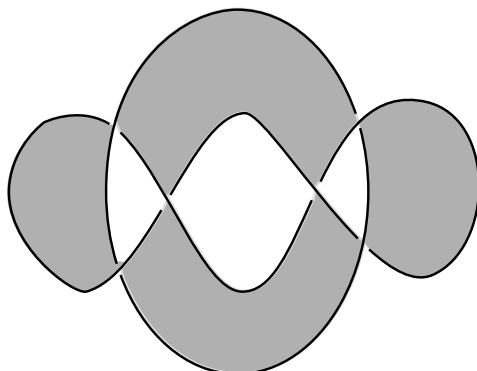


Figure 5.10: Non-orientable surface with  $\beta_1 = 3$

We also have the fact that the modified normal Euler number of  $F$  with the orientations above is  $-\{4\delta(A, A) + 2 + 4\delta(A, B) + 4\delta(B, A) + 4\delta(B, B)\}$ . Then since  $m = \delta(B, B)$ ,  $k = \delta(A, B)$  and  $n = \delta(A, A)$ , we obtain the second equality.  $\square$

We will prove the following conclusion.

**Theorem 5.4.3** *Let  $6_3^2$  denote the two-component link as illustrated in Figure 5.9, forgetting the orientations. We have  $\gamma(6_3^2) = 3$ .*

*Proof:* It is clear that the surface bounding the two-component  $6_3^2$  illustrated in Figure 5.10 is a non-orientable one with first Betti number 3.

Suppose that  $\gamma(6_3^2) \leq 2$ . Then there exists a non-orientable surface  $F$  bounding the link with  $H_1(F; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$ . The Goeritz matrix  $G_F(6_3^2)$  associated with  $F$  should be a 2 by 2 matrix, determined by a generator system of  $H_1(F; \mathbf{Z})$ . A different choice of basis of  $H_1(F; \mathbf{Z})$  gives another matrix  $J$  such that  $J = P^T G P$  where  $P$  is an integral unimodular matrix. The integral congruent class which  $G_F(6_3^2)$  belongs to does not change.

Since the link  $6_3^2$  is the two-bridge link  $S(12, 5)$ , then the double branched cover  $D(6_3^2)$  of  $S^3$  branched over  $6_3^2$  is the lens space  $L(12, 5)$  with  $H_1(D(6_3^2); \mathbf{Z}) =$

$\mathbf{Z}/12\mathbf{Z}$  and the linking form  $\lambda(g, g) = \pm\frac{5}{12}$  for some properly chosen generator of  $H_1(D(6_3^2); \mathbf{Z})$ . The determinant of the Goeritz matrix  $G_F(6_3^2)$  is known to be  $\pm 12$  since the absolute value of the determinant of a Goeritz matrix of a link is equal to the order of  $H_1(D(L); \mathbf{Z})$ .

By applying an elementary theorem of integral binary quadratic forms (see, for example [30]), we enumerate all the congruent classes of 2 by 2 integral matrices with discriminant  $\pm 48$ . The result is as follows:

$$\begin{aligned} X_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 12 \end{pmatrix}, X_2 = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}, X_3 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, X_4 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \\ X_5 &= \begin{pmatrix} 1 & 0 \\ 0 & -12 \end{pmatrix}, X_6 = \begin{pmatrix} -1 & 0 \\ 0 & 12 \end{pmatrix}, X_7 = \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}, \\ X_8 &= \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix}, X_9 = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}, X_{10} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}. \end{aligned}$$

It is obvious that the matrices  $X_2, X_4, X_7$  and  $X_8$  cannot present the cyclic group  $\mathbf{Z}/12\mathbf{Z}$ , and therefore cannot be the relation matrices of  $H_1(D(6_3^2); \mathbf{Z})$ . On the other hand, only the matrix  $X_3$  presents the linking form  $\pm\frac{5}{12}$  for the link  $6_3^2$ , see Chapter 2.

If  $X_3$  were a Goeritz matrix for the link  $6_3^2$ , then there should exist some integral unimodular matrix  $Q$  such that  $Q^T G_F(L) Q = X_3$ . The inverse matrix of  $Q$  is also integral unimodular and we denote it by  $P$ . Changing the basis of  $H_1(D(6_3^2); \mathbf{Z})$  by using  $P := \begin{pmatrix} r & u \\ s & v \end{pmatrix}$ , the Goeritz matrix  $G_F(L)$

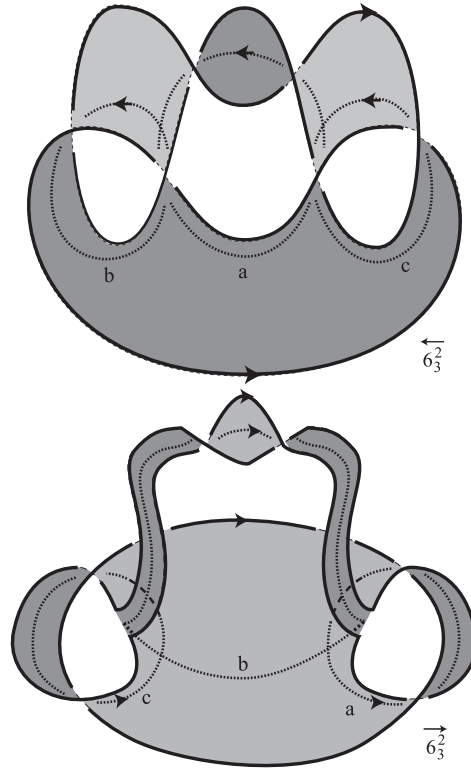


Figure 5.11: Seifert Surfaces with Generator Systems

is of the form:

$$P^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} P = \begin{pmatrix} 3r^2 + 4s^2 & 3ru + 4sv \\ 3ru + 4sv & 3u^2 + 4v^2 \end{pmatrix}.$$

Recall that the formula by Gordon and Litherland [12], also see Chapter 2, relates the signature of a link to the signature of the Goeritz matrix,  $\sigma(\bar{L}) = \sigma(G_F) + \frac{1}{2}\bar{e}(F)$ . The signature of a link is defined as the signature of the symmetrized Seifert matrix, the difference between the number of positive eigenvalues and negative ones of the matrix.

For some appropriate Seifert surfaces, see Figure 5.11, the corresponding Seifert matrices of the two-component link under two different orientations,  $\overleftarrow{6}_3^2$  and  $\overrightarrow{6}_3^2$ , see Figure 5.9, are as follows:

$$V(\overleftarrow{6}_3^2) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, V(\overrightarrow{6}_3^2) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then we have

$$\sigma(\overleftarrow{6}_3^2) = \sigma(V(\overleftarrow{6}_3^2) + V(\overleftarrow{6}_3^2)^T) = \sigma \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = 3,$$

and

$$\sigma(\overrightarrow{6}_3^2) = \sigma(V(\overrightarrow{6}_3^2) + V(\overrightarrow{6}_3^2)^T) = \sigma \begin{pmatrix} -2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -2 \end{pmatrix} = -1.$$

Namely, we obtain the signatures of  $\overleftarrow{6}_3^2$  and  $\overrightarrow{6}_3^2$  as 3 and  $-1$  respectively.

We will see that for either orientation, there exists no solution for which the formula by Gordon and Litherland holds.

According to Lemma 5.4.2, we see that the modified normal Euler number  $\bar{e}(F)$  is  $-2[3(r+u)^2 + 4(s+v)^2]$ . Now since  $\sigma(X_3) = 2$  we have

$$\sigma(\overleftarrow{6}_3^2) = 3 = 2 - [3(r+u)^2 + 4(s+v)^2], \quad (5.6)$$

or

$$\sigma(\overrightarrow{6_3^2}) = -1 = 2 - [3(r+u)^2 + 4(s+v)^2]. \quad (5.7)$$

Since a negative number cannot be equal to the sum of perfect squares, obviously we know that there exists no integral solution of  $r, s, u, v$  to the equality (5.6).

For  $\overrightarrow{6_3^2}$ , we establish another equality in terms of the linking number by Lemma 5.4.2. We have

$$\text{lk}(\overrightarrow{6_3^2}) = 2 = \frac{3u^2 + 4v^2}{2} + 3ru + 4sv. \quad (5.8)$$

Then we combine the equalities (5.7) and (5.8) to get the following system of equations:

$$\begin{cases} 3(r+u)^2 + 4(s+v)^2 = 3, \\ 3u^2 + 4v^2 + 6ru + 8sv - 4 = 0. \end{cases}$$

Because the values of  $r, u, s$  and  $v$  are taken in  $\mathbf{Z}$ , from the first equation we have  $r+u = \pm 1$  and  $s+v = 0$ . Transform the second equation to the following:

$$3(u+r)^2 - 3r^2 + 4(s+v)^2 - 4s^2 - 4 = 0.$$

Putting the values of  $u+r$  and  $s+v$  into the transformed equation, we have  $3 - 3r^2 - 4s^2 - 4 = 0$ , i.e.,  $3r^2 + 4s^2 = -1$ , for which there exists no integral solution of  $r$  and  $s$ .

So it turns out that there exist no integral solutions for this system of equations, which contradicts our assumption. Namely, there exists no connected, non-orientable surface with first Betti number 2. Therefore, we have

$$\gamma(6_3^2) = 3. \quad \square$$

## 5.5 Problems yet to solve

The examples we have already calculated are still quite few. It is interesting to know whether the method we used can be applied to the research on some certain family of 2-component links.

The integral binary quadratic form applied in the study has been proved to be quite useful to determine the crosscap numbers for some 2-component links. But the limit of its use is very obvious since it could only tell us whether the crosscap number of a 2-component link is 2 or not. For links with possibly bigger crosscap numbers, it may not be able to help us. We have been interested in finding a new way to study crosscap numbers, for example, to study the geometrical structures of 3-manifold. Namely, it is interesting for us to construct some non-orientable surfaces bounding the link which can possibly realize the minimum of first Betti numbers.

In [14], Hirasawa and Teragaito show that the crosscap number of a 2-bridge knot can be theoretically obtained as follows:

Generate all incompressible, boundary-incompressible surfaces bounding a two-bridge knot  $K$  according to [13]. Let  $n$  be the minimal first Betti number of them. If  $n$  is realized by a non-orientable surface, then  $\gamma(K) = n$ ; otherwise  $\gamma(K) = n + 1$ .

A similar scheme is expected to be applied to calculate the crosscap numbers of two-bridge links. For a 2-bridge link  $L_{p/q}$ , Floyd and Hatcher in [5] gave us a space of incompressible surfaces in the link complement

$S^3 - L_{p/q}$ . Encouraged by the results on crosscap numbers of 2-bridge knots, one may consider this space of incompressible surfaces to be related to the non-orientable surfaces realizing the minimum first Betti number for 2-bridge links, so that the crosscap numbers for 2-bridge links, a family of two-component links, could be hopefully known.

## Chapter 6

# Concordance crosscap number of a knot

We work again in the smooth category in this chapter, see [41]. Let  $K$  be a knot embedded in the three-sphere  $S^3$ . It can bound various orientable or non-orientable surfaces in  $S^3$  or in the four-ball  $B^4$  with  $\partial B^4 = S^3$ .

For orientable surfaces, one studies the genus. The *genus*  $g(K)$  of a knot  $K$  is the minimum genus among all the orientable surfaces that  $K$  bounds in  $S^3$  and the *slice genus*  $g^*(K)$  is the minimum genus of orientable surfaces that  $K$  bounds in  $B^4$ . From the viewpoint of the concordance group, one can define the concordance genus  $g_c(K)$  [21] as the minimum genus in the concordance class of  $K$ . It is easy to see that  $g^*(K) \leq g_c(K) \leq g(K)$ . In response to the question asked by Gordon [10], Casson in unpublished work and Nakanishi in [29] provided examples for which  $g^*(K) < g_c(K)$ . The relations among these knot invariants are also investigated by Livingston in [21].

For non-orientable surfaces, the first Betti number is taken as an invariant instead of the genus. The *crosscap number*  $\gamma(K)$ , also known as the *non-orientable genus*, of  $K$  is defined to be the minimum first Betti number of non-orientable surfaces in  $S^3$  bounding  $K$ . See [2, 14, 15, 24, 38] for studies on this invariant. The *4-dimensional crosscap number*  $\gamma^*(K)$  is the minimum first Betti number of non-orientable surfaces in the 4-ball  $B^4$  bounding  $K$ . Some results on 4-dimensional crosscap number by Murakami and Yasuhara can be found in [25].

In this chapter, we define the *concordance crosscap number*  $\gamma_c(K)$  as the minimum crosscap number among all the knots concordant to  $K$ , and study the gap between  $\gamma^*(K)$  and  $\gamma_c(K)$ . Note that by convention the crosscap number of the unknot is defined to be 0 for completeness. So if the knot  $K$  is a slice knot, we define  $\gamma^*(K) = \gamma_c(K) = 0$ .

We give a necessary condition for pretzel knots of type  $P(4-p, p, 2n-p)$  and  $P(-1-p, p, 2n-p)$  to have concordant crosscap number 1, and construct infinitely many knots  $K$  with  $\gamma^*(K) < \gamma_c(K)$ . In particular, the knot  $7_4$  is one of the examples.

## 6.1 Preliminaries

Two knots  $K_0$  and  $K_1$  in the 3-sphere  $S^3$  are called *concordant* if they cobound an annulus in  $S^3 \times [0, 1]$  with  $K_0$  and  $K_1$  in  $S^3 \times \{0\}$  and  $S^3 \times \{1\}$ , respectively. Concordance is an equivalence relation and the set of the equivalence classes forms an abelian group, the so-called *concordance group*. The *concordance crosscap number* is defined from this point of view.

Given a Seifert surface  $F$  bounding  $K$ , one can associate a Seifert matrix  $V$  to it. The *signature*  $\sigma(K)$  of  $K$  is defined to be the signature of the symmetric matrix  $V + V^T$ , i.e.  $\sigma(K) = \text{sign}(V + V^T)$ . It is well known that the knot signature is a concordance invariant [27].

Given a surface bounding a knot, not necessarily orientable, one can associate a Goeritz matrix with it (see, for example, [20]). Gordon and Litherland in [12] gave a simple algorithm to calculate the knot signature, expressed as the signature of the Goeritz matrix corresponding to the knot plus a correction term.

**Lemma 6.1.1** ([12]) *Let  $F$  be any surface bounding a knot  $K$ . Then the signature  $\sigma(K)$  can be calculated out of the Goeritz matrix  $G_F$  and the normal Euler number  $e(F)$ . Namely we have*

$$\sigma(K) = \text{sign}(G_F) + \frac{1}{2}e(F),$$

where  $e(F) := -\text{lk}(K, K')$ . Here  $K' := \partial N(K) \cap F$  and  $N(K)$  denotes the regular neighborhood of the knot  $K$ .

Note that the normal Euler number vanishes for an orientable surface  $F$ . In this case, the result conforms to the original definition.

In [2] Clark introduced the crosscap number, a knot invariant from a non-orientable viewpoint. The following result is well known.

**Lemma 6.1.2** ([2]) *A knot  $K$  has crosscap number 1 if and only if it is a  $(2, n)$ -cable knot.*

**Definition 6.1.3** *The concordance crosscap number of a knot  $K$ , denoted by  $\gamma_c(K)$ , is the minimum crosscap number among all the knots in the same concordance class as  $K$ .*

The 4-dimensional crosscap number, or non-orientable 4-genus [25],  $\gamma^*(K)$  is the minimum first Betti number of non-orientable surfaces in  $B^4$  bounding the knot  $K$ . The concordance crosscap number can be regarded as a bridge to relate the 4-dimensional invariant and the 3-dimensional one.

Clearly,  $\gamma^*(K) \leq \gamma_c(K) \leq \gamma(K)$ , and the inequalities are best possible.

**Proposition 6.1.4** *Let  $K$  be a  $(2, p)$ -cable knot with  $p \neq \pm 1$ . Then  $\gamma^*(K) = \gamma_c(K) = \gamma(K) = 1$ .*

*Proof:* It is known that  $\gamma(K) = 1$  by Lemma 6.1.2, and we will prove that  $\gamma^*(K) \neq 0$ . Suppose that  $K$  is the  $(2, p)$ -cable of a knot. Then it bounds a twisted Möbius band, whose corresponding Goeritz matrix is the 1-by-1 matrix  $\begin{pmatrix} p \end{pmatrix}$ . Its normal Euler number is known to be  $-2p$ , due to the fact that each positive crossing on the non-orientable band contributes  $-2$  to the normal Euler number. By Lemma 6.1.1 we have  $\sigma(K) = \text{sign}(p) - p$ . Then  $\sigma(K) \neq 0$  since  $p \neq \pm 1$ , and  $K$  is not a slice knot. Then  $\gamma^*(K) \geq 1$  and the proof is finished.  $\square$

Then we ask whether some knot exists such that all knots in its concordance class have crosscap numbers bigger than the 4-dimensional crosscap number of  $K$ , i.e. whether there exists a knot  $K$  with  $\gamma^*(K) < \gamma_c(K)$ .

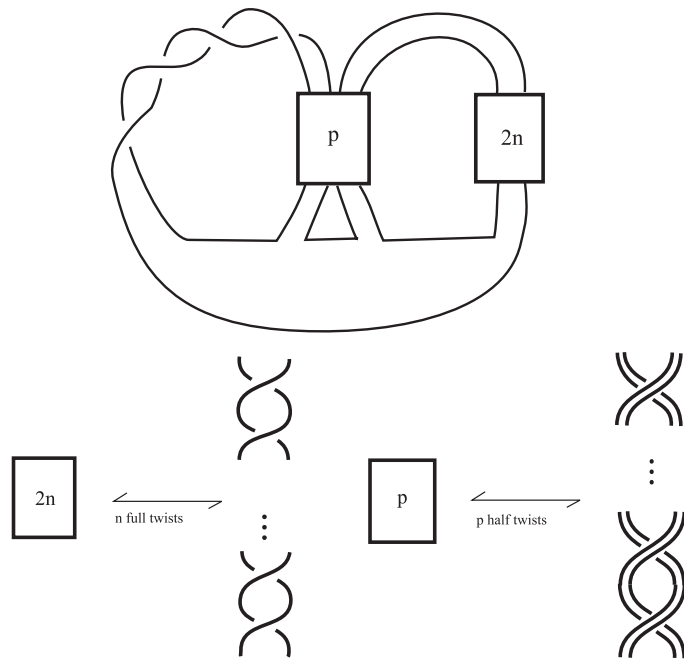


Figure 6.1:  $K(4, 2n, p)$

## 6.2 Examples constructed from Seifert surfaces

Consider the knot illustrated in Figure 6.1. We denote it by  $K(4, 2n, p)$ , where  $p$  is an odd number and  $n \in \mathbf{Z}$  denotes the number of the full twists on the right band. It is easy to see that the knot diagram actually represents the pretzel knot  $P(4 - p, p, 2n - p)$ .

By adding a twisted band as illustrated in Figure 6.2, the knot  $K(4, 2n, p)$  is changed into a ribbon knot. To see this, consider the indicated fission band in Figure 6.3, which gives an unlink. Reversing the fission process, one will see a fusion band intersecting two disks bounding the unlink in arcs. Since the ribbon knot is a slice knot [33], the disk in  $B^4$  that it bounds and the



Figure 6.2: Untwisted by adding a twisted band

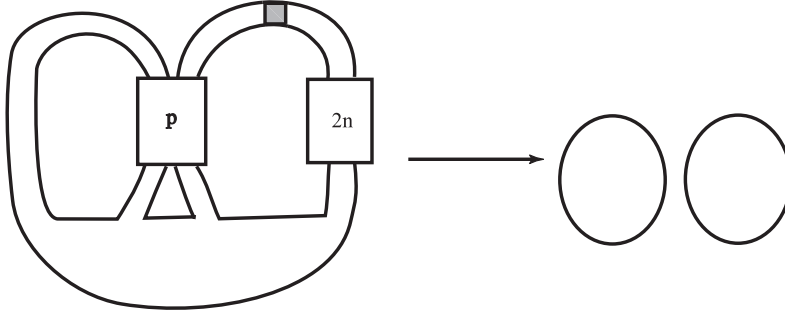


Figure 6.3: Fission band giving an unlink

twisted band together give a first Betti number 1 non-orientable surface in  $B^4$  bounding the knot  $K(4, 2n, p)$ , and we immediately have  $\gamma^*(K(4, 2n, p)) \leq 1$ .

**Theorem 6.2.1** *If  $\gamma_c(K(4, 2n, p)) = 1$ , there exists some odd number  $l$  such that the following equalities hold:*

$$\begin{cases} p^2 - 8n = l^2 & \text{when } 8n - p^2 < 0, \\ 3(8n - p^2) = l^2 & \text{when } 8n - p^2 > 0. \end{cases}$$

*Proof:* The Goeritz matrix  $G_{F_{p,n}}$  of the knot  $K(4, 2n, p)$  with respect to the Seifert surface  $F_{p,n}$  under a generator system  $\{a, b\}$ , indicated with oriented dashed lines in Figure 6.4, of the homology group  $H_1(F_{p,n}; \mathbf{Z})$  is of the form  $\begin{pmatrix} 4 & -p \\ -p & 2n \end{pmatrix}$ . Since the surface  $F_{p,n}$  is orientable, the normal Euler number  $e(F_{p,n})$  is 0 and the signature of  $K(4, 2n, p)$  is just the signature of the Goeritz matrix  $G_{F_{p,n}}$  according to Lemma 6.1.1.

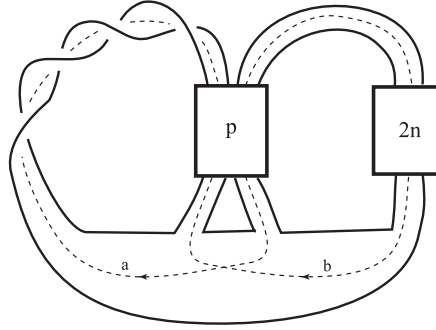


Figure 6.4: Generator system  $\{a, b\}$

Note that the value of  $\text{sign } G_{F_{p,n}}$  is determined by the sign of the  $2 \times 2$  matrix's determinant, i.e.,  $\text{sign } G_{F_{p,n}} = 0$  when  $\det G_{F_{p,n}} < 0$ , and  $\text{sign } G_{F_{p,n}} = 2$  when  $\det G_{F_{p,n}} > 0$ .

When  $\gamma_c(K(4, 2n, p)) = 1$ , there exists a knot  $K'$  concordant to  $K(4, 2n, p)$  with  $\gamma(K') = 1$ . So  $K'$  is a  $(2, q)$ -cable of some knot by Lemma 6.1.2, and  $\sigma(K') = \text{sign}(q) - q$  by the proof of Proposition 6.1.4. Due to the fact that the signature is a concordance invariant, we also have  $\sigma(K(4, 2n, p)) = \sigma(K')$ . Then by the above discussion  $\sigma(K') = \text{sign}(q) - q \in \{2, 0\}$  and by a calculation we obtain the following values:

$$\begin{cases} |q| = 1 & \text{when } \det G_{F_{p,n}} < 0, \\ q = -3 & \text{when } \det G_{F_{p,n}} > 0. \end{cases}$$

On the other hand, by concordance it is well known that the knot  $K(4, 2n, p)\sharp(-\overline{K'})$  is a slice knot where  $-\overline{K'}$  denotes the mirror image of  $K'$  with a reverse orientation. Then we have the Alexander polynomial  $\Delta(K(4, 2n, p)\sharp(-\overline{K'})) = \Delta(K, 2n, p)\Delta(K') = f(t)f(t^{-1})$  for some polynomial  $f(t)$  with  $f(1) = 1$  (see [6]). It follows that  $\det(K')\det(K(4, 2n, p)) = l^2$  for some odd number  $l$ ,

where  $\det(K)$ , the *determinant* of the knot  $K$ , is equal to  $|\Delta_K(-1)|$ . It is also known to be the order of the first homology group of double branched cover of  $S^3$  branched over  $K$ , and so we have  $\det(K(4, 2n, p)) = |\det G_{F_{p,n}}| = |8n - p^2|$ . Thus  $|8n - p^2| \cdot |q| = l^2$  and, precisely speaking,

$$\begin{cases} p^2 - 8n = l^2 & \text{when } 8n - p^2 < 0, \\ 3(8n - p^2) = l^2 & \text{when } 8n - p^2 > 0. \end{cases}$$

□

**Corollary 6.2.2** *For any odd number  $p$ , there exist infinitely many  $n$  for which the concordance crosscap number of the knot  $K(4, 2n, p)$  is greater than 1 and the 4-dimensional crosscap number is equal to 1.*

*Proof:* In the case of  $8n - p^2 > 0$ , we have  $\sigma(K(4, 2n, p)) = 2$ , and so the knot  $K(4, 2n, p)$  is not slice. Then we have  $\gamma^*(K(4, 2n, p)) = 1$ , and  $\gamma_c(K(4, 2n, p)) \geq 1$ . By Theorem 6.2.1 for a knot with  $\gamma_c(K(4, p, 2n)) = 1$  and  $8n - p^2 > 0$ , the equality  $3(8n - p^2) = l^2$  holds. Then the number 3 either becomes a common divisor of  $8n$  and  $p$ , or divides neither  $8n$  nor  $p$ . There exist infinitely many  $n$  with 3 dividing only one of  $8n$  and  $p$ , in which cases  $3(8n - p^2)$  cannot be a square for those  $n$ , and therefore a contradiction arises. The result follows. □

Therefore Corollary 6.2.2 gives examples demonstrating the gap between the 4-dimensional crosscap number and the concordance crosscap number. One can also discuss the knot of type  $K(-4, 2n, p)$  with reverse twists on the left band in the same way. Note that the crosscap number of the pretzel

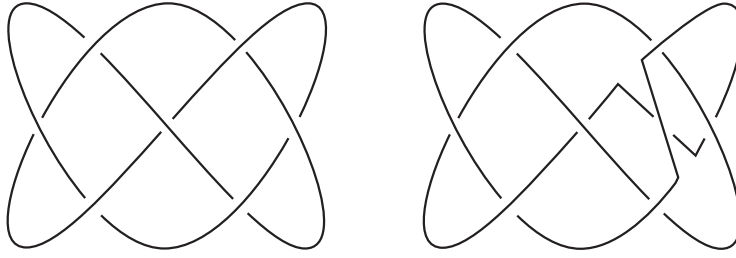


Figure 6.5: Knot  $7_4$  and unknotting of  $7_4$  by adding a twisted band

knot  $P(i_1, i_2, \dots, i_n)$  with all  $i_k$  odd is shown to be  $n$  in [15]. Then we know  $\gamma(K(4, 2n, p)) = \gamma(K(-4, 2n, p)) = 3$ , but we cannot say whether the concordance crosscap number of the knot  $K(4, 2n, p)$  (or  $K(-4, 2n, p)$ ) is 3 or not.

**Remark 6.2.3** *The knot  $K(-4, -4, -1) = P(-3, -1, -3)$  is knot  $7_4$  in Alexander-Briggs' notation [1, 33].*

### 6.3 Examples constructed from non-orientable surfaces

Consider the knot  $K(-1, 2n, p)$  illustrated in Figure 6.6, which is a pretzel knot of type  $P(-1 - p, p, 2n - p)$ . Here the  $n$  and  $p$  are the same as in Section 3, but in order to make our statement clearer and more succinct we only discuss the case with  $n > 0$ . The knot  $K(1, 2n, p)$  is the mirror image of  $K(-1, 2n, p)$  and thus the analysis of these knots follows from the analysis of  $K(-1, 2n, p)$ . It is obvious that  $\gamma^*(K(-1, 2n, p)) \leq 1$  since it can be unknotted by adding a half twisted band [19]. See Figure 6.7.

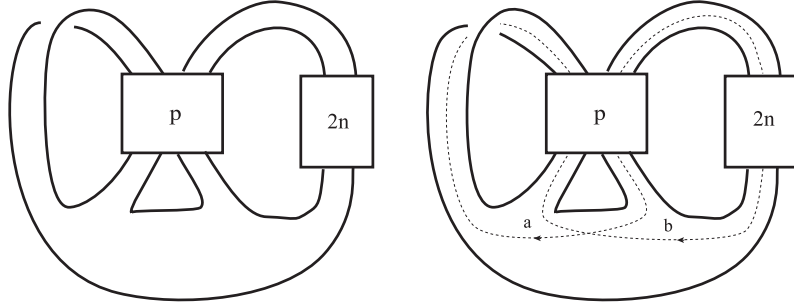


Figure 6.6:  $K(-1, 2n, p)$

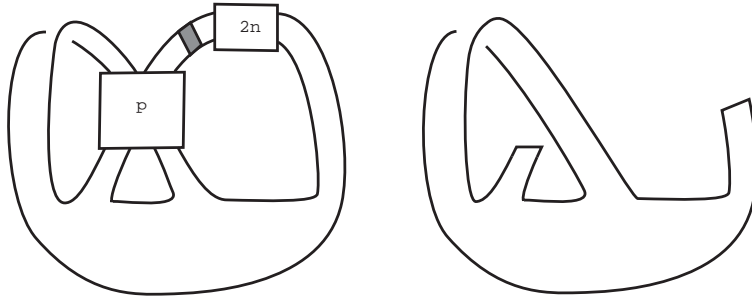


Figure 6.7: Unknotting by adding a twisted band

Note that the knot  $K(-1, 2n, \pm 1)$  is actually the torus knot  $T(2, 2n + 1)$  with  $\gamma^* = \gamma_c = \gamma = 1$  by Proposition 6.1.4.

**Theorem 6.3.1** *If  $\gamma_c(K(-1, 2n, p)) = 1$ , there exists an odd number  $l$  such that  $(2n + p^2)(2n + 1) = l^2$  when  $n > 0$ .*

*Proof:* The knot diagram bounds a non-orientable surface  $F'_{p,n}$  with normal Euler number  $-4n$ . The Goeritz matrix  $G_{F'_{p,n}}$ , corresponding to the non-orientable surface with the indicated basis  $\{a, b\}$  (see Figure 6.6) of the homology group  $H_1(F'_{p,n})$ , is of the form  $\begin{pmatrix} -1 & -p \\ -p & 2n \end{pmatrix}$ . Thus with  $n > 0$  we have  $\det G_{F'_{p,n}} = -2n - p^2 < 0$ , and hence  $\det K(-1, 2n, p) = 2n + p^2$ . By Lemma 6.1.1  $\sigma(K(-1, 2n, p)) = -2n$ , since  $\text{sign } G_{F'_{p,n}} = 0$  due to the negative sign of the matrix determinant.

If  $\gamma_c(K(-1, 2n, p)) = 1$ , through an argument similar to that for the knot of type  $K(4, 2n, p)$ , it is concordant to the  $(2, 2m + 1)$ -cable of a knot  $K'$ , with signature  $\sigma(K') = \text{sign}(2m + 1) - (2m + 1)$  and determinant  $\det(K') = 2m + 1$ . By concordance we have  $-2n = \text{sign}(2m + 1) - (2m + 1)$  which gives us the equality  $n = m$  by a calculation. Then the determinant of  $K'$  becomes  $2n + 1$ , and therefore there exists some odd number  $l$  such that  $\det K' \det K(-1, 2n, p) = (2n + 1)(2n + p^2) = l^2$  holds.  $\square$

**Remark 6.3.2** *Through the proof of Theorem 6.3.1 we know that the signature of the knot  $K(-1, 2n, p)$  is  $-2n$ , and so a different value of  $n$  gives us different concordance class of the knot type  $K(-1, 2n, p)$ . Therefore we can cover as many concordance classes as necessary.*

**Corollary 6.3.3** *For any odd number  $p \neq \pm 1$  there exist infinitely many  $n$  for which the concordance crosscap number of the knot  $K(-1, 2n, p)$  is greater than 1 and the 4-dimensional crosscap number is equal to 1.*

*Proof:* By Theorem 6.3.1 with  $n > 0$  we have  $(2n + p^2)(2n + 1) = l^2$  if  $\gamma_c(K(-1, 2n, p)) = 1$ . In this case  $\sigma(K(-1, 2n, p)) < 0$ , thus the knot  $K(-1, 2n, p)$  is not slice and  $\gamma^*(K(-1, 2n, p)) = 1$ . It is well known that infinitely many primes are in the set  $\{2m + 1 | 2m + 1 > p^2\} = \{2n + p^2 | n \in \mathbf{Z}^+\}$ . (It also immediately follows from Dirichlet's Theorem on Primes in Arithmetic Progressions [4].) It is clear that  $2n + 1$  cannot be divided by  $2n + p^2$ , which confirms that those  $(2n + p^2)(2n + 1)$  with prime  $2n + p^2$  cannot be a square. Therefore for the knots  $K(-1, 2n, p)$  with those  $n$  the concordance crosscap numbers are greater than 1.  $\square$

**Remark 6.3.4** *In [15] the pretzel knot  $P(i_1, i_2, \dots, i_n)$  with one  $i_k$  even and others odd is shown to have crosscap number  $n - 1$ . So for the knots in Corollary 6.3.3, we have  $\gamma_c(K(-1, 2n, p)) = 2$  since  $\gamma(P(-1 - p, p, 2n - p)) = 2$  for any  $(p, n)$ .*

In conclusion there are also numerous knots of type  $K(-1, 2n, p)$  with concordance crosscap number larger than their 4-dimensional crosscap number.

## 6.4 Further development by others

Inspired by the work above, Livingston [23] observes that the obstruction to  $\gamma_c(K) \leq 1$  found above extends to one based on the Alexander polynomial  $\Delta(K)$  of  $K$  and the signature  $\sigma(K)$ .

**Theorem 6.4.1** ([23]) *Suppose  $\gamma_c(K) = 1$  and set  $q = |\sigma(K)| + 1$ . For all odd prime power divisors  $p$  of  $q$ , the  $2p$ -cyclotomic polynomial  $\phi_{2p}(t)$  has odd exponent in  $\Delta_K(t)$ . Furthermore, every other symmetric irreducible polynomial  $\delta(t)$  with odd exponent in  $\Delta_K(t)$  satisfies  $\Delta(-1) = \pm 1$ .*

As a corollary, a necessary condition for the  $(p, q, r)$ -pretzel knots to have concordance crosscap number 1 is given as follows.

**Corollary 6.4.2** ([23]) *If  $\gamma_c(P(p, q, r)) = 1$ , then either  $\sigma(P(p, q, r)) = 0$  and  $pq + qr + rp = -l^2$  for some integer  $l$  or  $\sigma(P(p, q, r)) = \pm 2$  and  $pq + qr + rp = 3$ .*

Some of the knots with 4-dimensional crosscap number  $\gamma_4(K) = 1$  studied above are included. However, the construction of an obstruction to  $\gamma_c(K) \leq 2$  is still not available, and determining the concordance crosscap number for a knot is yet to be done.

# Bibliography

- [1] J. W. Alexander and G. B. Briggs, *On types of knotted curves*, Ann. of Math. (2) **28** (1926/27), no. 1-4, 562–586. MR 1502807
- [2] B. E. Clark, *Crosscaps and knots*, Internat. J. Math. Math. Sci. **1** (1978), no. 1, 113–123. MR 0478131 (57 #17620)
- [3] R. Crowell, *Genus of alternating link types*, Ann. of Math. (2) **69** (1959), 258–275. MR 0099665 (20 #6103b)
- [4] G. Everest and T. Ward, *An introduction to number theory*, Graduate Texts in Mathematics, vol. 232, Springer-Verlag London Ltd., London, 2005. MR 2135478
- [5] W. Floyd and A. Hatcher, *The space of incompressible surfaces in a 2-bridge link complement*, Trans. Amer. Math. Soc. **305** (1988), no. 2, 575–599. MR 924770 (89c:57004)
- [6] R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. **3** (1966), 257–267. MR 0211392 (35 #2273)

- [7] D. Gabai, *Foliations and genera of links*, *Topology* **23** (1984), no. 4, 381–394. MR 780731 (86h:57006)
- [8] ———, *Genera of the arborescent links*, *Mem. Amer. Math. Soc.* **59** (1986), no. 339, i–viii and 1–98. MR 823442 (87h:57010)
- [9] L. Goeritz, *Knoten und quadratische Formen*, *Math. Z.* **36** (1933), no. 1, 647–654. MR 1545364
- [10] C. McA. Gordon, *Problems*, *Knot theory* (Proc. Sem., Plans-sur-Bex, 1977), *Lecture Notes in Math.*, vol. 685, Springer, Berlin, 1978, pp. 309–311.
- [11] ———, *On the  $G$ -signature theorem in dimension four*, *À la recherche de la topologie perdue* (L. Guillou and A. Martin, eds.), *Progr. Math.*, vol. 62, Birkhäuser Boston, Boston, MA, 1986, pp. 159–180. MR 900251
- [12] C. McA. Gordon and R. A. Litherland, *On the signature of a link*, *Invent. Math.* **47** (1978), no. 1, 53–69. MR 0500905 (58 #18407)
- [13] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, *Invent. Math.* **79** (1985), no. 2, 225–246. MR 778125 (86g:57003)
- [14] M. Hirasawa and M. Teragaito, *Crosscap numbers of 2-bridge knots*, *Topology* **45** (2006), no. 3, 513–530. MR 2218754
- [15] K. Ichihara and S. Mizushima, *Crosscap numbers of pretzel knots*, arXiv:math.GT/0608480.

- [16] L. H. Kauffman, *Branched coverings, open books and knot periodicity*, *Topology* **13** (1974), 143–160. MR 0375337 (51 #11532)
- [17] L. H. Kauffman and L. R. Taylor, *Signature of links*, *Trans. Amer. Math. Soc.* **216** (1976), 351–365. MR 0388373 (52 #9210)
- [18] Y. Kawada, *Number Theory (in Japanese)*, Iwanami Shoten, 1992.
- [19] W. B. R. Lickorish, *Unknotting by adding a twisted band*, *Bull. London Math. Soc.* **18** (1986), no. 6, 613–615. MR 859958 (88e:57008)
- [20] ———, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978 (98f:57015)
- [21] C. Livingston, *The concordance genus of knots*, *Algebr. Geom. Topol.* **4** (2004), 1–22 (electronic). MR 2031909 (2005e:57023)
- [22] ———, *A survey of classical knot concordance*, *Handbook of knot theory*, Elsevier B. V., Amsterdam, 2005, pp. 319–347. MR 2179265 (2006k:57020)
- [23] ———, *Concordance crosscap number of knots and Alexander polynomial*, *Proc. Amer. Math. Soc.* **136** (2008), 3351.
- [24] H. Murakami and A. Yasuhara, *Crosscap number of a knot*, *Pacific J. Math.* **171** (1995), no. 1, 261–273. MR 1362987 (96k:57006)
- [25] ———, *Four-genus and four-dimensional clasp number of a knot*, *Proc. Amer. Math. Soc.* **128** (2000), no. 12, 3693–3699. MR 1690998 (2001b:57020)

- [26] K. Murasugi, *On the genus of the alternating knot. I, II*, J. Math. Soc. Japan **10** (1958), 94–105, 235–248. MR 0099664 (20 #6103a)
- [27] ———, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422. MR 0171275 (30 #1506)
- [28] ———, *On the signature of links*, Topology **9** (1970), 283–298. MR 0261585 (41 #6198)
- [29] Y. Nakanishi, *A note on unknotting number*, Math. Sem. Notes Kobe Univ. **9** (1981), no. 1, 99–108. MR 634000 (83d:57005)
- [30] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley & Sons Inc., New York, 1991. MR 1083765 (91i:11001)
- [31] P. Ozsváth and Z. Szabó, *Knot Floer homology, genus bounds, and mutation*, Topology Appl. **141** (2004), no. 1-3, 59–85. MR 2058681 (2005b:57028)
- [32] J.A. Rasmussen, *Khovanov homology and the slice genus*, arXiv:math.GT/0402131, to appear in *Inventiones Mathematicae*.
- [33] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990. MR 1277811 (95c:57018)
- [34] H. Schubert, *Knoten und Vollringe*, Acta Math. **90** (1953), 131–286. MR 0072482 (17,291d)

- [35] ———, *Knoten mit zwei Brücken*, Math. Z. **65** (1956), 133–170. MR 0082104 (18,498e)
- [36] ———, *Bestimmung der Primfaktorzerlegung von Verkettungen*, Math. Z. **76** (1961), 116–148. MR MR0141107 (25 #4519b)
- [37] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110** (1935), no. 1, 571–592. MR 1512955
- [38] M. Teragaito, *Crosscap numbers of torus knots*, Topology Appl. **138** (2004), no. 1-3, 219–238. MR 2035482 (2005a:57007)
- [39] A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264. MR 0248854 (40 #2104)
- [40] H. F. Trotter, *Homology of group systems with applications to knot theory*, Ann. of Math. (2) **76** (1962), 464–498. MR 0143201 (26 #761)
- [41] G. Zhang, *Concordance crosscap number of a knot*, Bull. Lond. Math. Soc. **39** (2007), no. 5, 755–761. MR 2365224 (2008k:57020)
- [42] ———, *Crosscap numbers of two-component links*, Kyungpook Math. J. **48** (2008), no. 2, 241–254. MR 2429312